

## NON-ARCHIMEDEAN WHITNEY-STRATIFICATIONS

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ABSTRACT. We define an analogue of Whitney stratifications for Henselian valued fields  $K$  of equi-characteristic 0 and prove that such stratifications exist. This analogue is a pretty strong notion; in particular, it sees singularities both at the level of the valued field and of the residue field. Using methods from non-standard analysis, we show how a stratification in our sense can be turned into a classical Whitney stratification of a given (semi-)algebraic subset of  $\mathbb{R}^n$  or  $\mathbb{C}^n$ .

As in the classical setting, we can work with different classes of subsets of  $K^n$ , e.g. algebraic sub-varieties or certain classes of analytic subsets. The general framework are definable sets (in the sense of model theory) in a language which satisfies certain hypotheses.

Another point of view is that our result describes sets up to ultra-metric isometry. In a previous article, a conjectural such description has been given for definable subsets of  $\mathbb{Z}_p^n$ ; the present result implies that conjecture when  $p$  is sufficiently big.

## 1. INTRODUCTION

A very useful tool in real and complex algebraic and analytic geometry are Whitney stratifications; see e.g. [11], [1]. In [2], it has been proven that Whitney stratifications also exist in the  $p$ -adics. The present article takes a different approach to transfer Whitney stratifications into a non-Archimedean setting; in a certain sense, our condition on the stratifications is much stronger than the usual ones. This allows us to show that if we work in a well-chosen valued field with residue field  $\mathbb{R}$  or  $\mathbb{C}$ , then a stratification satisfying our condition induces a classical Whitney stratification on the residue field.

We will mainly use the language of model theory, but we will give algebraic formulations of the most important results; the introduction is also supposed to be readable by non-model theorists.

We start by fixing some notation. Let  $K$  be a Henselian valued field of equi-characteristic 0 (i.e., both,  $K$  and its residue field have characteristic 0) and let us fix a suitable class  $\mathcal{C}$  of subsets of  $K^n$ . The precise requirements on  $\mathcal{C}$  will be given in Section 2; one can for example take  $\mathcal{C}$  to be the sub-varieties of  $K^n$  (not necessarily irreducible; so “sub-variety” means: locally closed in the Zariski topology), or definable subsets in a suitable language, in the sense of model theory. In particular, there are suitable languages including analytic subsets of  $K^n$ .

The goal is to understand the “singular locus” of sets  $X \in \mathcal{C}$ . Roughly, our main theorem states that given such a set  $X$ ,  $K^n$  can be partitioned into subsets  $S_0, \dots, S_d \in \mathcal{C}$  with  $\dim S_d = d$  such that at any point  $x \in S_d$ ,  $X$  is “non-singular

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in  $d$  directions”. More precisely, we obtain that on a suitable ball  $B$  around  $x$ , the family of sets  $(S_d, S_{d+1}, \dots, S_n, X)$  is “ $d$ -translatable”, which is defined as follows.

First, we have to introduce “risometries”, which play a central role in this article. To define them, we need a bit of notation. We write  $\mathcal{O}_K$  for the valuation ring of  $K$ ,  $k$  for the residue field,  $\Gamma$  for the value group,  $v: K \rightarrow \Gamma$  for the valuation map, and for  $x = (x_1, \dots, x_n) \in K^n$ , we define  $v(x) := \inf_i v(x_i)$ , i.e., we use the “maximum norm” on  $K^n$ ; when we speak of a ball in  $K^n$ , we mean ball with respect to this maximum norm. A risometry is something intermediate between an isometry (which preserves the valuation of differences) and a translation (which completely preserves differences): for a subset  $B \subseteq K^n$ , a map  $\phi: B \rightarrow B$  is a risometry iff it preserves differences up to the leading term:

$$v((\phi(y) - \phi(y')) - (y - y')) > v(y - y')$$

for all  $y, y' \in B$ .

We call the family of sets  $(S_d, S_{d+1}, \dots, S_n, X)$   $d$ -translatable on a ball  $B \subseteq K^n$  if there exists a definable (in the sense of model theory) risometry  $\phi: B \rightarrow B$  and a  $d$ -dimensional vector space  $\tilde{V} \subseteq K^n$  such that each set  $\phi(S_d \cap B), \dots, \phi(S_n \cap B), \phi(X \cap B)$  is translation invariant in direction  $\tilde{V}$ , i.e., it is the intersection of  $B$  with a union of cosets of  $\tilde{V}$ .

Using this, we can formulate a first version of the main theorem.

**Theorem 1.1.** *For every set  $X \subseteq K^n$  in our class  $\mathcal{C}$ , there exists a “ $t$ -stratification of  $K^n$  reflecting  $X$ ”, i.e., a partition  $(S_i)_{0 \leq i \leq n}$  of  $K^n$  with  $S_i \in \mathcal{C}$  such that for each  $d \leq n$ , we have the following:*

- $\dim S_d = d$  or  $S_d = \emptyset$
- For any ball  $B \subseteq S_d \cup \dots \cup S_n$ , the family  $(S_d, \dots, S_n, X)$  is  $d$ -translatable on  $B$ .

The “full version” of this theorem (formulated in the language of model theory) is Theorem 4.10. Corollary 4.11 is a reformulation which is uniform in the field  $K$  and which also works in sufficiently large positive characteristic. For readers not familiar with the language of model theory, Theorem 5.10 is an algebraic reformulation of Corollary 4.11.

In contrast to classical Whitney stratifications, the conditions on  $t$ -stratifications are not purely local, since we prescribe the size of the balls  $B$  where we require  $d$ -translatability. This makes it much stronger, but it also has some counter-intuitive implications; see Section 8 for examples.

For now, let us get some local intuition about  $t$ -stratifications and see how this relates to classical Whitney stratifications (their definition is recalled in Subsection 6.2). Fix a point  $x$  in some stratum  $S_d$ . First of all, it is not hard to deduce from the definition of  $t$ -stratifications that  $S_0 \cup \dots \cup S_{d-1}$  is topologically closed (Lemma 3.17), so we can find a ball  $B$  around  $x$  which is contained in  $S_d \cup \dots \cup S_n$ . By definition, we have  $d$ -translatability on  $B$ , i.e., we get a  $d$ -dimensional vector space  $\tilde{V} \subseteq K^n$  as explained above. In Whitney stratifications, each stratum is smooth, so that one can speak about tangent spaces. With  $t$ -stratifications, we do not know whether actual tangent spaces exist, but  $\tilde{V}$  can be seen as an “approximative tangent space” of  $S_d$  at  $x$ . (Such an approximative tangent space  $\tilde{V}$  is “unique up to smaller terms”; see Subsection 3.1 for details.) Moreover, for any  $j \geq d$  and any  $y \in B \cap S_j$ ,  $d$ -translatability on  $B$  implies that there exists an

approximative tangent space of  $S_j$  at  $y$  which contains  $\tilde{V}$ . So formulated sloppily, we have: for any  $x \in S_d$ , any  $j \geq d$ , and any  $y \in S_j$  close enough to  $x$ ,  $T_y S_j$  approximately contains  $T_x S_d$ . If one replaces “approximately contains” by “contains in the limit for  $y \rightarrow x$ ”, then the result is essentially the classical Condition (a) of Whitney.

Containing approximately sounds like a weaker condition and indeed, t-stratifications do not necessarily satisfy the straight forward translation of the Whitney conditions to non-Archimedean fields (in contrast to the stratifications from [2]). However, from the point of view of non-standard analysis, it is exactly the right translation; the important thing there that the ball  $B$  can be taken big enough. As a consequence, if we let  $K$  be a non-standard model of  $\mathbb{R}$  or  $\mathbb{C}$  (i.e., a particular valued field whose residue field  $k$  is equal to  $\mathbb{R}$  or  $\mathbb{C}$ , respectively), then any t-stratification of  $K^n$  induces a stratification of  $k^n$  which satisfies Condition (a).

Using this method, we will prove (Theorem 6.11) that t-stratifications induce classical Whitney stratifications. For this, we also need our t-stratifications to satisfy a non-standard version of Whitney’s Condition (b). A priori, this is not true: one can construct a counter-example using a kind of non-Archimedean logarithmic spiral. However, the t-stratifications we are considering consist of sets in the class  $\mathcal{C}$ , and inside this class, such counter-examples are excluded by the following theorem. Using it, we deduce the non-standard Condition (b) in Corollary 6.6.

**Theorem 1.2.** *For every set  $X \subseteq K^n$  in our class  $\mathcal{C}$  and every  $x \in K^n$ , there exists a finite subset  $M_x \subseteq \Gamma$  of the value group such that for any  $y \in K^n$  with  $v(y - x) \notin M_x$  and any ball  $B$  containing  $y$  but not  $x$ ,  $X$  is “translatable on  $B$  in direction  $K \cdot (y - x)$ ”, i.e., there exists a definable isometry  $\phi: B \rightarrow B$  such that  $\phi(X \cap B)$  is translation invariant in direction  $K \cdot (y - x)$ .*

The “full version” of this theorem is Theorem 6.4. Whereas Theorem 1.1 only yields the existence of translatability, Theorem 1.2 is a strong result about its direction. Indeed, formulated sloppily, it implies that for any fixed  $x \in K^n$  and almost any  $y \in X$  (more precisely: for  $y \in X$  at almost any distance from  $x$ ), the approximative tangent space  $T_y X$  approximately contains the line  $K \cdot (y - x)$ . In this generality, this might sound surprising from the Archimedean point of view (but note that for  $y$  close to  $x$ , it already has a flavor of Whitney’s Condition (b)).

In the Archimedean setting, given a finite family of subsets of  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , one can find a single Whitney stratification which fits to all those sets. In valued fields, we can even treat “small” infinite families of sets simultaneously. Here, “small” is not in the sense of cardinality; instead, a family of sets is small if it is parametrized by (a product of) subsets of the residue field  $k$  and the value group  $\Gamma$ . (In contrast, a family parametrized by the valued field  $K$  would be large.) To make sense of this, one needs the language of model theory, i.e., the class  $\mathcal{C}$  should be a suitable class of definable sets.

Now let us consider the above results from a completely different point of view. Part of the original motivation for the present article was to understand sets  $X \in \mathcal{C}$  up to isometry. It turned out that to get useful results, one has to work with a stronger notion, namely with isometries, as defined above. Indeed, t-stratifications “describe” sets up to isometry: if we have a t-stratification  $(S_i)_i$  reflecting a set  $X \subseteq K^n$ , then this means that up to applying a suitable isometry  $K^n \rightarrow K^n$ ,  $X$  (and  $(S_i)_i$ ) are pretty simple in the sense that there are many balls on which

they are translation invariant in many directions. Here are some more precise facts underlining this point of view.

- Suppose that  $(S_i)_i$  is a t-stratification of  $K^n$  and that  $X \subseteq K^n$  is any set in  $\mathcal{C}$ . Then  $(S_i)_i$  reflects  $X$  if and only if each risometry preserving  $(S_i)_i$  also preserves  $X$  (Proposition 3.20).
- The main conjecture of [8] can be seen as a description of definable sets in  $\mathbb{Q}_p$  up to isometry. In Section 7, we will check that existence of t-stratifications implies that conjecture for  $p$  sufficiently big (Theorem 7.1).
- Suppose we have a uniform family of sets  $X_q \subseteq K^n$  in  $\mathcal{C}$ , parametrized by  $q \in Q$  and suppose we want to decide which of them are risometric. A priori, this is a difficult task. In model theoretic terms, the induced equivalence relation on  $Q$  is not definable in general. However, if we assume that each  $X_q$  comes equipped with a t-stratification  $(S_{i,q})_i$  and we ask that these t-stratifications are also respected by the risometries, then the equivalence relation on  $Q$  becomes definable (Proposition 3.23). Moreover, the risometry type of  $(X_q, (S_{i,q})_i)$  can be described by a “finite amount of data living only in  $k$  and  $\Gamma$  (and not in  $K$ )”. A slightly weaker but purely algebraic version of this statement is given in Corollary 5.11.

In fact, we get even more. For each risometry class, there exists a uniformly definable family of risometries between each two  $(X_q, (S_{i,q})_i)$ , which is compatible with respect to composition (also Proposition 3.23).

Using this, we will deduce that all risometries  $\phi$  appearing in the definition of a t-stratification can be defined uniformly (Corollary 3.26). In particular, this turns “being a t-stratification” into a first order property.

The proof of the main theorem will use model theoretic methods; in particular,  $\mathcal{C}$  will be the class of definable sets in a suitable language, the basic case being the one where the language is simply the pure language of valued fields  $\mathcal{L}_{\text{Hen}}$  (see the beginning of Section 2). In that case, we will show a posteriori that even when we start with an arbitrary definable set  $X$ , we can get a t-stratification  $(S_i)_i$  consisting only of varieties (Corollary 5.9); this is why Theorem 1.1 also works when  $\mathcal{C}$  is the class of sub-varieties of  $K^n$ .

Our results also hold in different expansions of the language  $\mathcal{L}_{\text{Hen}}$ . In [3], Cluckers and Lipshitz introduce a quite general notion of “Henselian valued fields with analytic structure”. It turns out that their results are almost exactly the prerequisites needed for the present article, so this is a good general context to work in. More precisely, our prerequisites are summarized in Hypothesis 2.8 (and additionally Hypothesis 6.1 for Theorem 1.2), which in particular hold in the setting of [3] if the field has equi-characteristic 0 (see Propositions 2.12 and 6.2).

Here is an overview over the article. We start by fixing notation and by specifying the general assumptions on the language of valued fields in Section 2; These assumptions are summarized in Hypothesis 2.8. We also introduce risometries (proving first properties) and colorings—a handy way to treat small (in the above sense) infinite families of subsets of  $K^n$ .

The main purpose of the next section is to define translatability and t-stratifications and to prove first properties. We also give several characterizations of what it means for a t-stratification to reflect a set  $X$  (Subsection 3.3) and we show how t-stratifications are useful to understand definable families of sets up to risometry

(Subsection 3.4); the latter will be an important ingredient to the proof of the main result.

The bulk of the proof of the main result (Theorem 4.10) is done in Section 4; a sketch of the proof is given at the beginning of that section; the last subsection contains some corollaries.

The remaining sections give some variants and applications, mostly under some additional assumptions. In Section 5, we show how to obtain t-stratifications such that for each  $d$ ,  $S_0 \cup \dots \cup S_d$  is closed in a suitable topology. In the pure valued field language, this can be applied to the Zariski topology, which yields the algebraic version of the main result.

In the next section, we show how our result implies the existence of classical Whitney stratifications. To this end, we first prove the valued field version of Whitney's Condition (b) (Theorem 6.4, Corollary 6.6); this needs an additional (very natural) hypothesis on the language we are using (Hypothesis 6.1).

Finally, we show how the present results imply the main conjecture of [8] about sets up to isometry in  $\mathbb{Q}_p$  for  $p \gg 0$  (Section 7), we give a few examples of t-stratifications (Section 8), and we list some open questions concerning enhancements of the main result (Section 9).

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## 2. THE SETTING

**2.1. Model theoretic notation.** By  $\mathcal{T}_{\text{Hen}}$ , we will denote the theory of Henselian valued fields of equi-characteristic 0 in a suitable language  $\mathcal{L}_{\text{Hen}}$ . In almost all of the article, we only care about the language up to interdefinability. However, at some places, we will have to care about which sorts we are using; we will make this precise in Definition 2.1.

In most of the article, we will not use  $\mathcal{T}_{\text{Hen}}$  and  $\mathcal{L}_{\text{Hen}}$  itself, but an expansion  $\mathcal{T}$  in a language  $\mathcal{L} \supseteq \mathcal{L}_{\text{Hen}}$  (which has the same sorts as  $\mathcal{L}_{\text{Hen}}$ ); the precise conditions on  $\mathcal{T}$  and  $\mathcal{L}$  are given in Subsection 2.4.

Unless specified otherwise, "definable" will always mean definable with parameters. There will be some results concerning  $\emptyset$ -definable sets (of the form: for some  $\emptyset$ -definable  $X$ , there exists a  $\emptyset$ -definable  $Y \dots$ ). Our general assumptions will always allow to add parameters to the language (see Remark 2.10), so the reason to write " $\emptyset$ -definable" is only to emphasize that  $Y$  is definable over the same parameters as  $X$ .

If  $(X_q)_{q \in Q}$  is a family of definable sets (or maps), then we write  $\ulcorner X_q \urcorner$  for a "code" for  $X_q$ : if  $X_q$  is defined by a formula  $\phi(x, q)$ , then there exists a definable

map  $f: Q \rightarrow Q'$  for some definable set  $Q'$  (possibly imaginary) and a formula  $\psi(x, y)$  such that  $\psi(x, f(q))$  also defines  $X_q$  and  $f(q)$  is a canonical parameter for  $X_q$ . We set  $\ulcorner X_q \urcorner := f(q)$ . (Of course, this involves some choices.) When we only say that another set is  $\ulcorner X_q \urcorner$ -definable, then of course the choice of the code doesn't matter; however, sometimes we really want to choose  $f$  and  $\psi$  as above.

**2.2. Valued fields sorts.** Most of the time, we will work in a fixed model of  $\mathcal{T}_{\text{Hen}}$ , for which we use the following notation:  $K$  is the valued field,  $\mathcal{O}_K$  is the valuation ring,  $\mathcal{M}_K$  is its maximal ideal of  $\mathcal{O}_K$ ,  $k$  is the residue field,  $\Gamma$  is the value group, and  $\text{RV} = \{0\} \cup K^\times / (1 + \mathcal{M}_K)$  is the “leading term structure”. We write  $v: K \rightarrow \Gamma \cup \{\infty\}$  for the valuation,  $\text{res}: \mathcal{O}_K \rightarrow k$  for the residue map and  $\text{rv}: K \rightarrow \text{RV}$  for the canonical map  $K^\times \rightarrow K^\times / (1 + \mathcal{M}_K)$ , extended to  $K$  by  $0 \mapsto 0$ . Moreover, we have a canonical map  $v_{\text{RV}}: \text{RV} \rightarrow \Gamma \cup \{\infty\}$  (satisfying  $v_{\text{RV}} \circ \text{rv} = v$ ).

Recall that there is a short exact sequence  $k^\times \hookrightarrow K^\times / (1 + \mathcal{M}_K) \twoheadrightarrow \Gamma$ , so  $\text{RV}$  “consists of” the residue field and the value group. Indeed, any angular component map  $\text{ac}: K \rightarrow k$  induces a splitting of  $\text{RV} \setminus \{0\}$  into a direct product  $k^\times \times \Gamma$ .

**Definition 2.1.** (1) Let  $\mathcal{L}_{\text{Hen}}$  be the language consisting of one sort  $K$  for the valued field with the ring language and all sorts  $\text{RV}^{\text{eq}}$ . More precisely, by  $\text{RV}^{\text{eq}}$ , we mean the sort  $\text{RV}$  (defined above) with the map  $\text{rv}: K \rightarrow \text{RV}$ , and for each  $\emptyset$ -definable  $X \subseteq \text{RV}^k$  and each  $\emptyset$ -definable equivalence relation on  $X$ , a sort  $X/\sim$  and the canonical map  $X \rightarrow X/\sim$ .  
(2) We will call  $K$  the *main sort* and  $\text{RV}^{\text{eq}}$  the *auxiliary sorts*. By an *auxiliary set/element*, we will mean a subset/element of an auxiliary sort.  
(3) Let  $\mathcal{T}_{\text{Hen}}$  be the theory of Henselian valued fields of equi-characteristic 0 in the language  $\mathcal{L}_{\text{Hen}}$ .

Notationally, we will often treat  $\text{RV}^{\text{eq}}$  as the union of all auxiliary sorts. In particular, by a “definable map  $\chi: K^n \rightarrow \text{RV}^{\text{eq}}$ ”, we mean a definable map whose target is an arbitrary auxiliary sort (and similarly for definable sets  $Q \subseteq \text{RV}^{\text{eq}}$ ).

Of course, the value group and the residue field are auxiliary sorts. Now let us also introduce names for higher dimensional analogues of some of the above sorts and maps. Concerning the valuation, recall from the introduction that we defined  $v: K^n \rightarrow \Gamma \cup \{\infty\}$ ,  $v(x_1, \dots, x_n) = \min_i v(x_i)$ . We also write  $\text{res}$  for the map  $\mathcal{O}_K^n \rightarrow k^n$ . The right higher dimensional analogue of  $\text{RV}$  is obtained by thinking of  $\text{rv}(x)$  as “fixing  $x \in K$  up to something smaller”. This leads to the following definition.

**Definition 2.2.** Set  $\text{RV}^{(n)} := K^n / \sim$  where  $x \sim y \iff v(x-y) > v(x) \vee x = y = 0$  and write  $\text{rv}: K^n \rightarrow \text{RV}^{(n)}$  for the canonical map.

Note that  $\text{RV}^{(1)}$  is just  $\text{RV}$  and that  $\text{RV}^{(n)}$  is interpretable in  $\text{RV}$  (so it is an auxiliary sort), since for  $(x_i)_i, (y_i)_i \in K^n$ , if  $\text{rv}(x_i) = \text{rv}(y_i)$  for all  $i$ , then  $\text{rv}((x_i)_i) = \text{rv}((y_i)_i)$ . Also, we obtain a canonical map  $v_{\text{RV}}: \text{RV}^{(n)} \rightarrow \Gamma \cup \{\infty\}$  satisfying  $v_{\text{RV}}(\text{rv}(x)) = v(x)$  for  $x \in K^n$ . Note that on  $K^n$ , both  $v$  and  $\text{rv}$  are coordinate independent in the following sense:

**Lemma 2.3.** *For any matrix  $M \in \text{GL}_n(\mathcal{O}_K)$  and any  $x \in K^n$ , we have  $v(Mx) = v(x)$ . Moreover,  $M$  induces a map  $\text{RV}^{(n)} \rightarrow \text{RV}^{(n)}$ , also denoted by  $M$ , which satisfies  $\text{rv}(Mx) = M \text{rv}(x)$ .*

*Proof.* One easily checks  $v(Mx) \geq v(x)$ ; applying the same to  $M^{-1}$ , we get equality. Now this implies that  $M$  respects the equivalence relation from Definition 2.2, which yields the second claim.  $\square$

**Definition 2.4.** If  $\tilde{V} \subseteq K^n$  is a vector space, then we simply write  $\text{res}(\tilde{V})$  for the corresponding sub-vector space of  $k^n$ , i.e.,  $\text{res}(\tilde{V}) = \{\text{res}(x) \mid x \in \tilde{V} \cap \mathcal{O}_K^n\}$ . Vice versa, if  $V \subseteq k^n$  is a vector space, then any vector space  $\tilde{V} \subseteq K^n$  with  $\text{res}(\tilde{V}) = V$  will be called a *lift* of  $V$ .

Finally, we will need a notion for the “leading term of the direction of a vector in  $K^n$ ”.

**Definition 2.5.** For  $x \in K^n \setminus \{0\}$ , let the *direction* of  $x$  be the one-dimensional subspace  $\text{dir}(x) := \text{res}(K \cdot x)$  of  $k^n$ , considered as an element of the projective space  $\mathbb{P}^n k$  (which is also an auxiliary sort). Notationally, we will almost always treat  $\text{dir}(x)$  as a representative  $v \in \text{res}(K \cdot x)$  of the actual direction. Whenever we will use this notation, we will make sure that the particular choice of  $v$  doesn't matter.

One easily verifies that the direction map factors over  $\text{RV}^{(n)}$ ; we write  $\text{dir}_{\text{RV}}$  for the corresponding map  $\text{RV}^{(n)} \rightarrow \mathbb{P}^n k$  (i.e.,  $\text{dir}_{\text{RV}} \circ \text{rv} = \text{dir}$ ).

Here are some basic properties of the above sorts and maps.

- Lemma 2.6.** (1)  $v: K^n \rightarrow \Gamma \cup \{\infty\}$  defines an ultra-metric on  $K^n$ , i.e.,  $v(a+b) \geq \min\{v(a), v(b)\}$ .
- (2) If  $a_1, a_2 \in K^n$  satisfy  $v(a_1 + a_2) = \min\{v(a_1), v(a_2)\}$ , then  $\text{rv}(a_1)$  and  $\text{rv}(a_2)$  together determine  $\text{rv}(a_1 + a_2)$ , i.e., for any other  $a'_1, a'_2 \in K^n$  with  $\text{rv}(a'_i) = \text{rv}(a_i)$ , we have  $\text{rv}(a'_1 + a'_2) = \text{rv}(a_1 + a_2)$ .
- (3) Suppose that  $\pi: K^n \twoheadrightarrow K^d$  is a coordinate projection,  $\bar{\pi}: k^n \twoheadrightarrow k^d$  is the corresponding projection at the level of the residue field, and  $a \in K^n \setminus \{0\}$ . Then we have  $v(\pi(a)) = v(a)$  iff  $\bar{\pi}(\text{dir}(a)) \neq 0$ . Moreover, in that case  $\bar{\pi}(\text{dir}(a)) = \text{dir}(\pi(a))$ , and if  $a' \in K^n$  is another element with  $\pi(a') = \pi(a)$  and  $\text{dir}(a') = \text{dir}(a)$ , then we have  $\text{rv}(a') = \text{rv}(a)$ .
- (4) Let  $\langle \cdot, \cdot \rangle$  denote the standard scalar product, both on  $K^n$  and on  $k^n$ . Then for  $a, b \in K^n$  we have  $v(\langle a, b \rangle) > v(a) + v(b)$  iff  $\langle \text{dir}(a), \text{dir}(b) \rangle = 0$ .

*Proof.* Easy. (It is often helpful to assume without loss that some element has valuation 0.)  $\square$

### 2.3. Balls, projections, and fibers.

- Definition 2.7.** (1) An *open ball* in  $K^n$  is a set of the form  $B(a, > \delta) := \{x \in K^n \mid v(x - a) > \delta\}$  for  $a \in K^n$  and  $\delta \in \Gamma \cup \{-\infty\}$ .
- (2) A *closed ball* is a set of the form  $B(a, \geq \delta) := \{x \in K^n \mid v(x - a) \geq \delta\}$  for  $a \in K^n$  and  $\delta \in \Gamma$ .
- (3) A *ball* is either an open or a closed ball.
- (4) The *radius* of a ball  $B$  is the above  $\delta$ ; we denote it by  $\text{rad}_o(B)$  if  $B$  is an open ball and by  $\text{rad}_c(B)$  if  $B$  is a closed ball.

Thus: we do consider  $K^n$  as a ball (an open one), but we do not consider points as balls, and neither do we allow arbitrary cuts in  $\Gamma$  as radii of balls. The reason to have two different notations  $\text{rad}_o$  and  $\text{rad}_c$  is that if  $\Gamma$  is discrete, then any ball  $B \neq K^n$  can be considered both as an open or as a closed ball and  $\text{rad}_o(B)$  is strictly bigger than  $\text{rad}_c(B)$ . Note also that since we are using the maximum norm,

a ball in  $K^n$  is a product of  $n$  balls in  $K$  of the same radius, so the word “cube” would also be appropriate.

The set of (open or closed or all) balls in  $K^n$  is parametrized by an auxiliary sort; we will feel free to use quantifiers over balls in first order formulas.

From time to time, given a ball  $B$  we will need to consider the ball  $B'$  of the same radius containing the origin. We do not introduce a special notation for this; instead, note that  $B' = B - B = \{b - b' \mid b, b' \in B\}$ .

We will work a lot with projections  $\pi: K^n \twoheadrightarrow K^d$  to some subset of the coordinates. The corresponding projection  $k^n \twoheadrightarrow k^d$  at the level of the residue field will be denoted by  $\bar{\pi}$ . By  $\pi^\vee: K^n \twoheadrightarrow K^{n-d}$ , we will denote the “complementary” coordinate projection of  $\pi$ , i.e.,  $(\pi, \pi^\vee): K^n \rightarrow K^d \times K^{n-d}$  is an isomorphism. Often, we will consider restricted coordinate projections  $\pi: B \rightarrow K^d$  for some subset  $B \subseteq K^n$  (most of the time, a ball); in that case,  $\bar{\pi}$  still denotes the entire projection  $k^n \twoheadrightarrow k^d$ .

Given a coordinate projection  $\pi: B \rightarrow K^n$ , any fiber  $\pi^{-1}(x)$  (for  $x \in \pi(B)$ ) can be identified with a subset of  $K^{n-d}$  via  $\pi^\vee$ . Using this, any definition made for  $K^{n-d}$  can also be applied to fibers of coordinate projections (for example, this yields a notion of a ball inside a fiber  $\pi^{-1}(x)$ ).

**2.4. Requirements on the theory.** Now let us fix the general assumptions of this article. In the simplest setting,  $K$  is just a model of  $\mathcal{T}_{\text{Hen}}$  in the language  $\mathcal{L}_{\text{Hen}}$  (see Definition 2.1). However, everything also works in any expansion of  $\mathcal{T}_{\text{Hen}}$  satisfying the following Hypothesis.

**Hypothesis 2.8.** Throughout the article, we will assume that  $\mathcal{T}$  is an expansion of  $\mathcal{T}_{\text{Hen}}$  in a language  $\mathcal{L} \supseteq \mathcal{L}_{\text{Hen}}$  with the following properties:

- (1) RV is stably embedded, i.e., any definable subset of  $\text{RV}^n$  is definable using only parameters from RV.
- (2) Definable maps from RV to  $K$  have finite image.
- (3)  $\mathcal{T}$  is b-minimal with centers and with weak 1-Jacobian property in the sense of Definition 2.9.

The notions in the third conditions have been introduced and described in [4] and [3]. B-minimality with centers is a list of axioms designed to yield a certain kind of cell decomposition and a notion of dimension. Jacobian property additionally imposes conditions on definable functions in one variable. Here are the definitions; they differ slightly from the ones in [3]; see below.

**Definition 2.9.** The expansion  $\mathcal{T}$  of  $\mathcal{T}_{\text{Hen}}$  is *b-minimal with centers* if for every model  $K \models \mathcal{T}$  and every set  $A \subseteq K \cup \text{RV}^{\text{eq}}$ , the following holds.

- (1) For any  $A$ -definable set  $X \subseteq K$ , there exists an  $A$ -definable auxiliary set  $Q \subseteq \text{RV}^{\text{eq}}$  and  $A$ -definable maps  $c: Q \rightarrow K$  and  $\xi: Q \rightarrow \text{RV}$  such that the family  $(c(q) + \text{rv}^{-1}(\xi(q)))_{q \in Q}$  is a partition of  $X$ . (Note that each set  $c(q) + \text{rv}^{-1}(\xi(q))$  is either an open ball or a point.)
- (2) There is no surjective  $A$ -definable map from an auxiliary set to a ball  $B \subseteq K$ .
- (3) For every  $A$ -definable  $X, Y \subseteq K$  and  $\phi: X \rightarrow Y$ , there exists an  $A$ -definable map  $\chi: X \twoheadrightarrow Q \subseteq \text{RV}^{\text{eq}}$  such that for each  $q \in Q$ ,  $\phi$  restricted to the fiber  $\chi^{-1}(q)$  is either injective or constant.

The theory  $\mathcal{T}$  moreover has the *weak 1-Jacobian property* if the following stronger version of (3) holds.

- (3') For  $X, Y, \phi$  as in (3),  $\chi: X \rightarrow Q \subseteq \text{RV}^{\text{eq}}$  can be chosen such that each of its fibers is either a point or an open ball, and for each fiber  $F = \chi^{-1}(q)$  which is a ball, there exists a  $\xi \in \text{RV}$  such that  $\text{rv}(\frac{\phi(x) - \phi(x')}{x - x'}) = \xi$  for all  $x, x' \in F, x \neq x'$ .

Note that condition (2) has only been included for completeness; anyway, it follows from Hypothesis 2.8 (2).

The differences between Definition 2.9 and the definitions in [3] are the following:

- In [3], the language consists only of the sorts  $K$  and  $\text{RV}$ , whereas we use  $K$  and all sorts of  $\text{RV}^{\text{eq}}$ . It is not clear to me whether the two-sorted version of the definitions implies the multi-sorted version. However, the proofs in [3] go through verbatim in the multi-sorted language; see Subsection 2.5 for some details.
- Our weak 1-Jacobian property differs from the Jacobian property of [3, Definition 6.3.5] in the “weak” and in the “1”. In [3, Definition 6.3.5], if one replaces the condition  $v(a) + v(x - x') = v(\phi(x) - \phi(x'))$  by  $\text{rv}(a) \cdot \text{rv}(x - x') = \text{rv}(\phi(x) - \phi(x'))$ , then one obtains what is called the 1-Jacobian property. The difference between this and what we call the weak 1-Jacobian property is that in the former one, one additionally requires  $\phi$  to be continuously differentiable on each fiber  $F$ .

**Remark 2.10.** Note that all conditions in Hypothesis 2.8 stay true if we add parameters to the language. (Since our theory  $\mathcal{T}$  is not complete, by “adding parameters” we simply mean adding constant symbols to the language.) In particular, any result proven for  $\emptyset$ -definable sets automatically also holds over any parameter set  $A$ . This will be used throughout the proofs without further mentioning.

**2.5. Analytic structures satisfy our hypothesis.** In [3], Cluckers and Lipshitz introduced a quite general notion of “valued fields with analytic structure” and they proved in particular that such fields are b-minimal with centers and have the Jacobian property. We will now check that these also satisfy Hypothesis 2.8, but first, let us consider some examples. The most basic analytic structure is the trivial one, i.e., the valued field  $K$  is simply a model of the theory  $\mathcal{T}_{\text{Hen}}$  in the language  $\mathcal{L}_{\text{Hen}}$ . (More precisely, in [3], the language only consists of the sorts  $K, \text{RV}$  and not all other sorts of  $\text{RV}^{\text{eq}}$ .) Here is a non-trivial example:

**Example 2.11.** Let  $A := \mathbb{Z}[[t]]$  be equipped with the  $t$ -adic valuation (which we denote by  $v$ ), let

$$T_m := A\langle \xi_1, \dots, \xi_m \rangle = \left\{ \sum_{\nu \in \mathbb{N}^m} c_\nu \xi^\nu \mid c_\nu \in A, \lim_{|\nu| \rightarrow \infty} v(c_\nu) = \infty \right\}$$

be the algebra of restricted power series (here, we use multi-index notation), and set

$$S_{m,n} := T_m[[\rho_1, \dots, \rho_n]].$$

As a language, take  $\mathcal{L} := \mathcal{L}_{\text{Hen}} \dot{\cup} \bigcup_{m,n} S_{m,n}$ , where each element of  $S_{m,n}$  is a symbol for an  $(m+n)$ -ary function.

Now suppose that  $K$  is a complete valued field of rank one extending  $A$  (the requirement that  $K$  extends  $A$  amounts to choosing an image for the element  $t$  of  $A$

in the maximal ideal  $\mathcal{M}_K$  of the valuation ring of  $K$ ). Then each element of  $S_{m,n}$  naturally defines a function  $\mathcal{O}_K^m \times \mathcal{M}_K^n \rightarrow \mathcal{O}_K$ . This turns  $K$  into an  $\mathcal{L}$ -structure (after extending these functions trivially to  $K^{m+n}$ ) and as such,  $K$  is a valued field with analytic structure in the sense of [3].

Note that for Hypothesis 2.8 to apply to  $K$ , we additionally need  $K$  to be of equi-characteristic 0. Concretely, one could take for example  $K = \mathbb{C}((t))$ .

Many other (more general) examples are given in [3, Subsection 4.4]. Now let us check that all this falls into the scope of the present article.

**Proposition 2.12.** *In valued fields with analytic structure in the sense of [3], Hypothesis 2.8 holds.*

*Proof.* In this proof, we will use the result [3, Theorem 6.3.7] about elimination of  $K$ -quantifiers. This uses a language  $\mathcal{L}_{\text{Hen},\mathcal{A}}$  similar to our  $\mathcal{L}$ , but with additional relations on  $\text{RV}$ . In  $\mathcal{L}_{\text{Hen},\mathcal{A}}$ , the only sorts are  $K$  and  $\text{RV}$ , but adding the sorts  $\text{RV}^{\text{eq}}$  to  $\mathcal{L}_{\text{Hen},\mathcal{A}}$  doesn't impede this kind of quantifier elimination.

Hypothesis 2.8 (1) follows from [3, Theorem 6.3.7], using that in  $\mathcal{L}_{\text{Hen},\mathcal{A}}$ , the only connection between  $K$  and  $\text{RV}$  is the map  $\text{rv}$ . Hypothesis 2.8 (2) follows from [3, Theorem 6.3.8], which describes definable functions into  $K$ . Hypothesis 2.8 (3) is (a part of) [3, Theorem 6.3.7], except for the differences mentioned after Definition 2.9; let us now look more closely at those differences.

By [3, Remark 6.3.16], one can get the 1-Jacobian property instead of just the Jacobian property, and our hypothesis is a weakening of that. Concerning the additional sorts of  $\text{RV}^{\text{eq}}$ , we have to check what happens if the parameter set  $A$  contains elements of  $\text{RV}^{\text{eq}}$ .

In Lemma 6.3.14 of [3], anyway one may suppose  $A \subseteq K$ ; this implies (1) of Definition 2.9 (called (b1) and “with centers” in [3]).

Definition 2.9 (2) anyway follows from Hypothesis 2.8 (2).

In [3], (b3) (i.e., (3) from Definition 2.9) is deduced using Lemma 2.4.4 of [4], by proving a statement denoted by “(\*)”, namely: not all fibers of a definable map from a subset of  $K$  to a ball  $B \subseteq K$  can contain balls. We get our variant of (b3) by applying [4, Lemma 2.4.4] to  $(K, \text{RV}^{\text{eq}})$  instead of  $(K, \text{RV})$ .

To adapt the proof of the Jacobian property to the language with  $\text{RV}^{\text{eq}}$ , it now suffices that in each place where b-minimality is applied in [3], we use the structure  $(K, \text{RV}^{\text{eq}})$  instead of  $(K, \text{RV})$ . On our way, we have to prove  $\text{RV}^{\text{eq}}$ -versions of [3, Theorem 6.3.8] and [3, Lemma 6.3.15]. These two results concern a specific language  $\mathcal{L}_{\text{Hen},\mathcal{A}}^*$  (expanding  $\mathcal{L}_{\text{Hen},\mathcal{A}}$ ) on  $(K, \text{RV})$ . To get a corresponding language on  $(K, \text{RV}^{\text{eq}})$ , we can simply add to  $\mathcal{L}_{\text{Hen},\mathcal{A}}^*$  the new sorts of  $\text{RV}^{\text{eq}}$  and the corresponding canonical maps. In fact, the precise language on the auxiliary sorts doesn't matter; the only important part of  $\mathcal{L}_{\text{Hen},\mathcal{A}}^*$  are the terms involving  $K$ .  $\square$

**2.6. First consequences: dimension and spherically completeness.** By [4], b-minimality implies the existence of a good notion of dimension of definable sets (which in particular satisfies the axioms given in [10]).

**Definition 2.13.** Let  $X \subseteq K^n$  be a definable set. The *dimension*  $\dim X$  is the maximal  $d$  such that there exists a coordinate projection  $\pi: K^n \twoheadrightarrow K^d$  such that  $\pi(X)$  contains a ball. We set  $\dim \emptyset := -\infty$ . For  $x \in K^n$ , the *local dimension* of  $X$  at  $x$  is  $\dim_x X := \min\{\dim(X \cap B(x, > \gamma)) \mid \gamma \in \Gamma\}$ .

It is clear that dimension is definable, i.e., if  $X_q \subseteq K^n$  is a  $\emptyset$ -definable family of sets (for  $q \in Q$ ), then  $\{q \in Q \mid \dim X_q = d\}$  is  $\emptyset$ -definable for every  $d$ . Moreover, we have the following.

**Lemma 2.14** ([4], [10]). *Dimension has the following properties:*

- (1) *If  $(X_q)_{q \in Q}$  is a definable family of subsets of  $K^n$  and  $Q \subseteq \text{RV}^{\text{eq}}$  is auxiliary, then  $\dim \bigcup_{q \in Q} X_q = \max_q \dim X_q$ .*
- (2) *If  $f: X \rightarrow Y$  is a definable map for some definable sets  $X \subseteq K^m, Y \subseteq K^n$  and if each fiber  $f^{-1}(y)$  has dimension  $d$ , then  $\dim X = \dim Y + d$ .*

By (1) (or directly by b-minimality), a subset of  $K^n$  is 0-dimensional iff it is the image of a map  $\text{RV}^{\text{eq}} \rightarrow K^n$ ; hence Hypothesis 2.8 (2) is equivalent to requiring that 0-dimensional sets are finite.

We will also need the following property of local dimension:

**Lemma 2.15.** *Let  $X \subseteq K^n$  be a definable set and set  $Y := \{x \in X \mid \dim_x X < \dim X\}$ . Then  $\dim Y < \dim X$ .*

Let me just sketch how this follows from Hypothesis 2.8. A “better” proof (using only general assumptions on the dimension) is given in the short note [6].

*Proof.* It suffices to prove that there is an  $y \in Y$  with  $\dim_y Y = \dim Y$ . Then  $\dim Y = \dim X$  would imply  $y \notin Y$  (by definition of  $Y$ ).

Set  $d := \dim Y$ . Using cell decomposition [4, Theorem 3.7] and up to permutation of coordinates, we find a definable function  $f$  from a  $d$ -dimensional ball  $B \subseteq K^d$  to  $K^{n-d}$  whose graph is contained in  $Y$ .

Using the Jacobian property in all directions, i.e., letting vary only one coordinate of the domain at a time (as in the proof of Lemma 4.3) and considering each coordinate of the range separately, we can assume that  $f$  is continuous on  $B$  (after shrinking it), and hence, for any  $z \in B$ ,  $Y$  has local dimension  $d$  at  $(z, f(z))$ .  $\square$

Hypothesis 2.8 also directly implies that  $K$  is “definable spherically complete”:

**Lemma 2.16.** *For every definable family  $(B_q)_{q \in Q}$  of balls  $B_q \subseteq K$  which form a chain with respect to inclusion, the intersection  $\bigcup_{q \in Q} B_q$  is non-empty.*

*Proof.* Let such a family  $(B_q)_{q \in Q}$  be given. We can assume without loss that  $q$  is the radius of  $B_q$ , i.e., in particular  $Q \subseteq \Gamma$ . We may suppose that  $Q$  has no maximum.

By b-minimality, we find sets  $T_q \subseteq \text{RV}^{\text{eq}}$  and maps  $f_q: T_q \rightarrow K$  and  $\xi_q: T_q \rightarrow \text{RV}$  such that

$$B_q = \bigcup_{t \in T_q} (f_q(t) + \text{rv}^{-1}(\xi_q(t)));$$

in particular  $B_q \subseteq B(f_q(t), \geq q)$  for every  $t \in T_q$ . This can be done uniformly in  $q$ , so  $(f_q)_q$  can be considered as a map from the auxiliary set  $\bigcup_{q \in Q} T_q$  to  $K$  which, by Hypothesis 2.8 (2), has finite image. Choose an element  $a \in K$  such that  $\{q \in Q \mid a \in \text{im } f_q\}$  is co-final in  $Q$ . Then  $B_q \subseteq B(a, \geq q)$  implies  $a \in B_{q'}$  for all  $q' < q$  and hence  $a \in \bigcap_q B_q$ .  $\square$

**2.7. Risometries.** Let us now have a look at the notion of risometry, which already appeared in the introduction. Recall the definition.

**Definition 2.17.** For  $X, Y \subseteq K^n$ , a *risometry* from  $X$  to  $Y$  is a bijection  $\phi: X \rightarrow Y$  satisfying  $\text{rv}(\phi(x) - \phi(x')) = \text{rv}(x - x')$  for any  $x, x' \in X$ .

As in  $\text{rv}$ , the “r” in “risometry” stands for “residue field”.

Since the map  $\text{rv}: K^n \rightarrow \text{RV}^{(n)}$  is compatible with linear maps  $M \in \text{GL}_n(\mathcal{O}_K)$  (see Lemma 2.3), such maps also preserve risometries: if  $B \subseteq K^n$  is a ball and  $\phi: B \rightarrow B$  is a risometry, then we also have a risometry  $M \circ \phi \circ M^{-1}: M(B) \rightarrow M(B)$ . This will be used from time to time to “without loss change coordinates”. More precisely, we can apply any coordinate transformation at the level of the residue field, since any matrix  $\bar{M} \in \text{GL}_n(k)$  can be lifted to a matrix  $M \in \text{GL}_n(\mathcal{O}_K)$ .

In spherically complete valued fields, there is no risometry from a ball  $B$  to a proper subset of  $B$ . For definable risometries, this follows from Lemma 2.16 about definable spherically completeness. The proof uses the following “definable Banach fixed point theorem”.

**Lemma 2.18.** *Let  $B \subseteq K^n$  be a ball and suppose that  $f: B \rightarrow B$  is definable and contracting in the sense that for any  $x_1, x_2 \in B$  with  $x_1 \neq x_2$ ,  $v(f(x_1) - f(x_2)) > v(x_1 - x_2)$ . Then  $f$  has (exactly) one fixed point.*

*Proof.* Suppose that  $f(x) \neq x$  for all  $x \in B$ . For  $x \in B$ , set

$$B_x := \{y \in B \mid v(y - x) \geq v(x - f(x))\}.$$

For two different points  $x, x' \in B$ , the assumption  $v(f(x) - f(x')) > v(x - x')$  implies  $v(x - x') \geq \min\{v(x - f(x)), v(x' - f(x'))\}$  and hence either  $B_x$  contains  $x'$  or vice versa. In particular,  $B_x \cap B_{x'} \neq \emptyset$ , so all balls  $B_x$  form a chain under inclusion and by Lemma 2.16, their intersection  $\bigcup_{x \in B} B_x$  contains an element  $x_0$ . However, again by assumption we have  $v(f(x_0) - f(f(x_0))) > v(x_0 - f(x_0))$  which implies  $x_0 \notin B_{f(x_0)}$ .  $\square$

**Lemma 2.19.** *Let  $B \subseteq K^n$  be a ball and let  $f: B \rightarrow X$  be a definable risometry with  $X \subseteq B$ . Then  $X = B$ .*

*Proof.* Let  $x_0 \in B$  be given; the idea is to find a preimage of  $x_0$  by Newton-approximation (although  $f$  might not be differentiable, it behaves as if the derivative would be approximately 1): given an approximation  $x \in B$  of a preimage of  $x_0$ , we define the next approximation to be  $g(x) := x + x_0 - f(x)$ . Obviously, a fixed point of  $g$  is a preimage of  $x_0$ , so we just need to verify that  $g$  is contracting. Indeed:

$$g(x_1) - g(x_2) = (x_1 - x_2) - (f(x_1) - f(x_2)).$$

Since  $f$  is a risometry,  $\text{rv}(x_1 - x_2) = \text{rv}(f(x_1) - f(x_2))$ , i.e.  $v(g(x_1) - g(x_2)) > v(x_1 - x_2)$ .  $\square$

Next, let us describe risometries between finite sets and how such risometries can be extended to larger sets. In the following, for  $x \in K^n$  and  $T \subseteq K^n$ , the notation  $\text{rv}(x - T)$  means  $\{\text{rv}(x - t) \mid t \in T\}$ .

**Lemma 2.20.** *Let  $T \subseteq K^n$  be a finite set.*

- (1) *The only risometry  $T \rightarrow T$  is the identity. (In particular, between two different finite sets, there is at most one risometry.)*
- (2) *For  $x_1, x_2 \in K^n$   $x_1 \neq x_2$ , the following are equivalent:*

- (a) *There exists a risometry  $\phi: K^n \rightarrow K^n$  with  $\phi(T) = T$  and  $\phi(x_1) = x_2$ .*
- (b)  $B(x_1, \geq v(x_1 - x_2)) \cap T = \emptyset$ .
- (c)  $\text{rv}(x_1 - T) = \text{rv}(x_2 - T)$ .
- (3) *A map  $\phi: K^n \rightarrow K^n$  which is the identity on  $T$  is a risometry if and only if for each maximal ball  $B \subseteq K^n \setminus T$ , the restriction  $\phi|_B$  is a risometry from  $B$  to itself.*

*Proof.* (2) “(a)  $\Rightarrow$  (c)” and “(b)  $\Rightarrow$  (a)” are trivial. (For the latter, define  $\phi$  to be the translation by  $x' - x$  on  $B := B(x_1, \geq v(x_1 - x_2))$  and the identity everywhere else.)

“(c)  $\Rightarrow$  (b)” : Without loss,  $v(x_1 - x_2) = 0$  and  $x_1, x_2 \in \mathcal{O}_K^n$ . Suppose for contradiction that  $T_0 := T \cap \mathcal{O}_K^n$  is non-empty. The assumption implies  $\text{rv}(x_1 - T_0) = \text{rv}(x_2 - T_0)$  and hence  $\text{res}(x_1 - T_0) = \text{res}(x_2 - T_0)$ . This implies

$$\sum_{\bar{t} \in \text{res}(T_0)} (\text{res}(x_1) - \bar{t}) = \sum_{\bar{t} \in \text{res}(T_0)} (\text{res}(x_2) - \bar{t}).$$

Adding  $\sum_{\bar{t} \in \text{res}(T_0)} \bar{t}$  and then dividing by  $|\text{res}(T_0)|$  on both sides yields  $\text{res}(x_1) = \text{res}(x_2)$ , which contradicts  $v(x_1 - x_2) = 0$ .

(1) If  $\phi: T \rightarrow T$  is a risometry, then for any  $t \in T$  we have  $\text{rv}(t - T) = \text{rv}(\phi(t) - \phi(T)) = \text{rv}(\phi(t) - T)$ . Suppose that  $\phi(t) \neq t$ . Then (2) “(c)  $\Rightarrow$  (b)” yields  $B(t, \geq v(t - \phi(t))) \cap T = \emptyset$ , which contradicts  $t \in T$ .

(3) “ $\Rightarrow$ ” follows from (2), (a)  $\Rightarrow$  (b). For “ $\Leftarrow$ ”, suppose that  $\phi|_B$  is a risometry  $B \rightarrow B$  for each maximal ball  $B \subseteq K^n \setminus T$ ; we have to verify that  $\text{rv}(x - x') = \text{rv}(\phi(x) - \phi(x'))$  for every  $x, x' \in K^n$ . If  $x$  and  $x'$  lie in the same maximal ball  $B$ , then there is nothing to show. Otherwise, we have  $v(x - x') < v(x - \phi(x))$  and  $v(\phi(x) - x') < v(x' - \phi(x'))$ , which implies  $\text{rv}(x - x') = \text{rv}(\phi(x) - x') = \text{rv}(\phi(x) - \phi(x'))$ .  $\square$

**2.8. Colorings.** Recall that our goal is to find, for a given definable set  $X \subseteq K^n$ , a “t-stratification reflecting  $X$ ”. In the introduction, we already mentioned that we will do this not only for a single set  $X$ , but also for a “small” definable family  $(X_q)_{q \in Q}$  of sets. The precise notion of small is that  $Q$  is auxiliary.

When we require a t-stratification  $(S_i)_i$  to reflect  $(X_q)_{q \in Q}$ , we do not just require  $(S_i)_i$  to reflect each set  $X_q$  individually (as defined in Theorem 1.1); instead, for any ball  $B \subseteq S_d \cup \dots \cup S_n$ , we require the family  $(S_d, \dots, S_n, (X_q)_{q \in Q})$  to be  $d$ -translatable as a whole. (However, later this will turn out not to make a difference; see Remark 3.21.) In other words, we require that the sets  $\phi(S_i \cap B)$  and  $\phi(X_q \cap B)$  are  $\tilde{V}$ -translation invariant for all  $i$  and  $q$ , where the risometry  $\phi: B \rightarrow B$  and the space  $\tilde{V} \subseteq K^n$  do not depend on the set. This means that to check whether  $(S_i)_i$  reflects  $(X_q)_{q \in Q}$ , the only thing one needs to know about  $(X_q)_{q \in Q}$  is the equivalence relation on  $K^n$  defined by

$$(*) \quad x \sim x' \iff \forall (q \in Q) (x \in X_q \leftrightarrow x' \in X_q).$$

Indeed, all  $\phi(X_q \cap B)$  are  $\tilde{V}$ -translation invariant iff  $\phi(Y \cap B)$  is  $\tilde{V}$ -translation invariant for each  $\sim$ -equivalence class  $Y$ .

From this point of view, instead of working with families  $(X_q)_{q \in Q}$ , we may as well work with the following kind of “colorings”, which will be more handy.

**Definition 2.21.** A *coloring* of a definable set  $X \subseteq K^n$  is a definable map  $\chi: X \rightarrow \text{RV}^{\text{eq}}$ . Fibers of a coloring will be called *monochromatic pieces*. For two subsets

$Y_1, Y_2 \subseteq X$ , we will say that a map  $\phi: Y_1 \rightarrow Y_2$  *respects*  $\chi$  if  $\chi|_{Y_2} \circ \phi = \chi|_{Y_1}$ . When a coloring  $\chi: X \rightarrow \text{RV}^{\text{eq}}$  is given, we will often consider subsets  $Y_i \subseteq X$  as colored sets, i.e., we implicitly require maps  $Y_1 \rightarrow Y_2$  to respect  $\chi$ . In particular,  $Y_1$  and  $Y_2$  will be called *risometric* (or *risometric when colored with  $\chi$* , in case of ambiguity) if there exists a risometry  $Y_1 \rightarrow Y_2$  respecting  $\chi$ .

We say that another coloring  $\chi': X \rightarrow \text{RV}^{\text{eq}}$  is a *refinement* of  $\chi$  if  $\chi'(x_1) = \chi'(x_2)$  implies  $\chi(x_1) = \chi(x_2)$  for any  $x_1, x_2 \in X$ . The *product* of two colorings  $\chi, \chi': X \rightarrow \text{RV}^{\text{eq}}$  is the map  $x \mapsto (\chi(x), \chi'(x))$ ; we denote it by  $(\chi, \chi')$ .

Any coloring of a definable set  $X$  induces the equivalence relation “having the same color” on  $X$ . The following lemma states that for our purposes, colorings and small definable families of sets are the same.

**Lemma 2.22.** *Let  $X \subseteq K^n$  be definable. The set of equivalence classes on  $X$  obtained from  $\emptyset$ -definable colorings of  $X$  is the same as the set of equivalence classes on  $X$  obtained from  $\emptyset$ -definable families  $(Y_q)_{q \in Q}$  as above in (\*).*

*Proof.* Given a coloring  $\chi$ , we take the family of sets consisting of the monochromatic pieces of  $\chi$ . Vice versa, if a family  $(Y_q)_{q \in Q}$  of subsets of  $X$  is given, where  $Q \subseteq \text{RV}^{\text{eq}}$  is auxiliary, then each  $x \in X$  yields an  $x$ -definable auxiliary set  $Q_x := \{q \in Q \mid x \in Y_q\}$ . By stably embeddedness of  $\text{RV}$ , we can find a code  $\ulcorner Q_x \urcorner$  in  $\text{RV}^{\text{eq}}$ . The coloring  $\chi: x \mapsto \ulcorner Q_x \urcorner$  has the desired properties.  $\square$

**Convention 2.23.** Using this lemma, we will feel free to treat families of definable subsets of  $K^n$  as colorings. In particular, this will quite often be applied to definable partitions  $(S_i)_{i \leq n}$  of  $K^n$ . Frequently, we will moreover have a coloring  $\chi: K^n \rightarrow \text{RV}^{\text{eq}}$ ; then  $((S_i)_i, \chi)$  denotes the product of the coloring corresponding to  $(S_i)_i$  and  $\chi$ .

### 3. T-STRATIFICATIONS

In this section, we make the definition of t-stratification more precise and we prove a bunch of basic properties. We start by looking more closely at the notion of translatability.

**3.1. Translatability.** Recall that a lift of a sub-space  $V \subseteq k^n$  is any sub-space  $\tilde{V} \subseteq K^n$  with  $\text{res}(\tilde{V}) = V$ .

**Definition 3.1.** Let  $B \subseteq K^n$  be a ball and  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  a coloring.

- (1) For a sub-space  $\tilde{V} \subseteq K^n$ , we say that  $\chi$  is  *$\tilde{V}$ -translation invariant* (on  $B$ ) if for any  $x, x' \in B$  with  $x - x' \in \tilde{V}$ , we have  $\chi(x) = \chi(x')$ .
- (2) For a sub-space  $V \subseteq k^n$ , we say that  $\chi$  is  *$V$ -translatable* (on  $B$ ) if there exists a lift  $\tilde{V} \subseteq K^n$  of  $V$  and a definable risometry  $\phi: B \rightarrow B$  such that  $\chi \circ \phi$  is  $\tilde{V}$ -translation invariant on  $B$ ;  $\phi$  will be called a *straightener* (of  $\chi$  on  $B$ ).
- (3) For an integer  $d \in \{0, \dots, n\}$ , we say that  $\chi$  is  *$d$ -translatable* (on  $B$ ) if there exists a  $d$ -dimensional  $V \subseteq k^n$  such that  $\chi$  is  $V$ -translatable.

It will turn out that for the notion of  $V$ -translatability, the choice of the lift  $\tilde{V}$  doesn't matter. However, before we prove this, we introduce another important notion, namely a certain kind of “transversality” to  $\tilde{V}$ . More precisely, in the following definition one should think of the fibers of the projection  $\pi$  as being “sufficiently transversal” to any lift  $\tilde{V}$  of the space  $V \subseteq k^n$ .

**Definition 3.2.** Let  $V \subseteq k^n$  be a sub-vector space. An *exhibition* of  $V$  is a coordinate projection  $\pi: K^n \rightarrow K^d$  inducing an isomorphism  $\bar{\pi}: V \xrightarrow{\sim} k^d$ . (In particular,  $d = \dim V$ .) We also say that  $\pi$  *exhibits*  $V$ . If  $B \subseteq K^n$  is a subset (usually a ball), then the restriction  $\pi|_B$  will also be called an exhibition of  $V$ .

Obviously, exhibitions exist for arbitrary  $V \subseteq k^n$ . One also easily checks that for  $\tilde{V} \subseteq K^n$ ,  $\pi: K^n \rightarrow K^d$  exhibits  $\text{res}(\tilde{V})$  iff  $v(x) = v(\pi(x))$  for all  $x \in \tilde{V}$ .

Now the following lemma says that the choice of  $\tilde{V}$  doesn't matter in Definition 3.1 (2).

**Lemma 3.3.** *Suppose that  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  is a  $V$ -translatable coloring. Then for any lift  $\tilde{V} \subseteq K^n$  of  $V$ , there exists a definable risometry  $\phi: B \rightarrow B$  such that  $\chi \circ \phi$  is  $\tilde{V}$ -translation invariant.*

*Proof.* By  $V$ -translatability, there exists a risometry  $\phi': B \rightarrow B$  such that  $\chi \circ \phi'$  is  $\tilde{V}'$ -translation invariant on  $B$  for some lift  $\tilde{V}' \subseteq K^n$  of  $V$ . By replacing  $\chi$  with  $\chi \circ \phi'$ , we may assume that  $\chi$  itself is  $\tilde{V}'$ -translation invariant on  $B$ . Let us moreover assume that  $0 \in B$ .

Choose an exhibition  $\pi: B \rightarrow K^d$  of  $V$  and define  $\phi: K^n \rightarrow K^n$  to be the linear map sending  $\tilde{V}$  to  $\tilde{V}'$  and satisfying  $\pi \circ \phi = \pi$ . Then  $\chi \circ \phi$  is  $\tilde{V}$ -translation invariant, so it remains to verify that  $\phi$  is a risometry. For this, it is enough to check that  $\text{rv}(x) = \text{rv}(\phi(x))$  for all  $x \in K^n$ .

Without loss,  $v(x) = 0$ ; in that case, what we have to show is  $\text{res}(x) = \text{res}(\phi(x))$ . Write  $x = y + z$  with  $y \in \tilde{V}$  and  $\pi(z) = 0$ . The fact that  $\pi$  is an exhibition of  $V$  implies  $y, z \in \mathcal{O}_K^n$ . Now  $\text{res}(\phi(y)) = \text{res}(y)$  and  $\phi(z) = z$  together imply  $\text{res}(x) = \text{res}(\phi(x))$ , which finishes the proof.  $\square$

It is clear that if a coloring  $\chi$  is  $V$ -translatable on a ball  $B$ , then it is also  $V'$ -translatable on  $B$  for any sub-space  $V' \subseteq V$ . Also, since risometries preserve balls,  $V$ -translatability on  $B$  implies  $V$ -translatability on  $B'$  for any sub-ball  $B' \subseteq B$ . A slightly less obvious fact is the following.

**Lemma 3.4.** *Suppose that a coloring  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  is both,  $V_1$  and  $V_2$ -translatable for some  $V_1, V_2 \subseteq k^n$ . Then  $\chi$  is  $(V_1 + V_2)$ -translatable.*

*Proof.* Without loss,  $V_1 \cap V_2 = 0$  and  $0 \in B$ . Moreover, using Lemma 2.3 we may assume that  $V_1 = k^d \times \{0\}^{n-d}$  and  $V_2 \subseteq \{0\}^d \times k^{n-d}$ . Choose lifts  $\tilde{V}_1 := K^d \times \{0\}^{n-d}$  and  $\tilde{V}_2 \subseteq \{0\}^d \times K^{n-d}$  of  $V_1$  and  $V_2$ , respectively.

Let us write elements of  $B$  as  $(x, y)$  for  $x \in K^d$  and  $y \in K^{n-d}$ .

Without loss,  $\chi$  is  $\tilde{V}_1$ -translation invariant, i.e.,  $\chi(x, y) = \chi(x', y)$  for any  $x, x', y$ . Let  $\phi: B \rightarrow B$  be a risometry such that  $\chi \circ \phi$  is  $\tilde{V}_2$ -translation invariant, and define  $\psi: B \rightarrow B$  by  $\psi(x, y) := \phi(0, y) + x$ . We claim that  $\psi$  is a risometry and that  $\chi \circ \psi$  is  $(V_1 + V_2)$ -translation invariant.

Consider  $(x_1, y_1), (x_2, y_2) \in B$ . We have  $\text{rv}(\psi(0, y_1) - \psi(0, y_2)) = \text{rv}(0, y_1 - y_2)$ , so using Lemma 2.6 (2), we get

$$\text{rv}(\psi(0, y_1) - \psi(0, y_2) + (x_1 - x_2, 0)) = \text{rv}((0, y_1 - y_2) + (x_1 - x_2, 0)),$$

which implies that  $\psi$  is a risometry.

Now suppose that  $y_1 - y_2 \in V_2$ . Then  $\chi(\psi(x_1, y_1)) = \chi(\phi(0, y_1) + x_1) = \chi(\phi(0, y_1)) = \chi(\phi(0, y_2)) = \chi(\phi(0, y_2) + x_2) = \chi(\psi(x_2, y_2))$ .  $\square$

By the previous lemma, for every coloring and every ball, there exists a (unique) maximal space in which the coloring is translatable on that ball; let us name it.

**Definition 3.5.** Let  $\chi: B_0 \rightarrow \text{RV}^{\text{eq}}$  is a coloring and  $d \in \mathbb{N}$ . For any sub-ball  $B \subseteq B_0$ , we define the *translation space* of  $\chi$  on  $B$  to be the maximal sub-space  $\text{tsp}_B(\chi) \subseteq k^n$  such that  $\chi$  is  $\text{tsp}_B(\chi)$ -translatable on  $B$ .

Using this definition, we have:  $\chi$  is  $V$ -translatable on  $B$  iff  $V \subseteq \text{tsp}_B(\chi)$ , and  $\chi$  is  $d$ -translatable iff  $\dim \text{tsp}_B(\chi) \geq d$ .

For many arguments concerning a  $V$ -translatable coloring  $\chi$ , we will work on fibers of an exhibition  $\pi$  of  $V$ . The following lemma summarizes the basic facts needed for this.

**Lemma 3.6.** *Let  $B \subseteq K^n$  be a ball, let  $\tilde{V} \subseteq K^n$  be a sub-vector space and let  $\pi: B \rightarrow K^d$  be an exhibition of  $V := \text{res}(\tilde{V})$ .*

- (1) *If  $\phi: B \rightarrow B$  a definable risometry, then the unique map  $\phi': B \rightarrow B$  satisfying  $\pi \circ \phi' = \pi$  and  $\phi'(z) - \phi(z) \in \tilde{V}$  for all  $z \in B$  is a risometry.*

*In particular, if  $\chi$  is a  $V$ -translatable coloring on  $B$ , we have the following:*

- (2) *There exists a definable risometry  $\phi: B \rightarrow B$  satisfying  $\pi \circ \phi = \pi$  such that  $\chi \circ \phi$  is  $\tilde{V}$ -translation invariant. (In other words,  $\phi$  is a straightener respecting the fibers of  $\pi$ .)*
- (3) *For any definable risometry  $\psi: B \rightarrow B$  and any  $\pi$ -fiber  $\pi^{-1}(x) \subseteq B$  (for  $x \in \pi(B)$ ), there exists a definable risometry  $\psi': F \rightarrow F$  such that  $(\chi \circ \psi)|_F = (\chi|_F) \circ \psi'$ .*

In (3), one can think of  $\chi$  and  $\chi \circ \psi$  as two different but risometric colorings; from that point of view, the conclusion is that the restrictions of these two colorings to a  $\pi$ -fiber are also risometric.

*Proof of Lemma 3.6.* (1) Without loss,  $\tilde{V} = K^d \times \{0\}^{n-d}$ , and  $\pi$  is the projection to the first  $d$  coordinates. Recall that  $\pi^\vee$  denotes the complementary projection, i.e., in this case the projection to the last  $n-d$  coordinates. Again, write an element of  $B$  as  $(x, y)$  with  $x \in K^d$ ,  $y \in K^{n-d}$ . In this notation, we have  $\phi'(x, y) = (x, \pi^\vee(\phi(x, y)))$ .

In general, if  $z, z', z'' \in K^n$  satisfy  $\pi(z') = \pi(z)$  and  $\pi^\vee(z') = \pi^\vee(z'')$ , then  $\text{rv}(z'') = \text{rv}(z)$  implies  $\text{rv}(z') = \text{rv}(z)$ . Apply this as follows: for  $(x_1, y_1), (x_2, y_2) \in B$ , set  $z := (x_1, y_1) - (x_2, y_2)$ ,  $z' := \phi'(x_1, y_1) - \phi'(x_2, y_2)$  and  $z'' := \phi(x_1, y_1) - \phi(x_2, y_2)$ .

(2) Let  $\psi$  be a straightener of  $\chi$  and apply (1) to its inverse  $\psi^{-1}$ . The inverse  $\phi$  of the resulting map is a straightener of  $\chi$  satisfying  $\pi \circ \phi = \pi$ .

(3) Let  $\phi$  be a straightener of  $\chi$  satisfying  $\pi \circ \phi = \pi$ . Applying (1) to  $\phi^{-1} \circ \psi$  yields a map  $\phi'$  satisfying  $\pi \circ \phi' = \pi$  and  $\chi \circ \psi = \chi \circ \phi \circ \phi'$ . Define  $\psi'$  to be the restriction of  $\phi \circ \phi'$  to  $F$ .  $\square$

Using this, we can give an alternative characterization of translatability. Recall that for a ball  $B$ ,  $B - B$  is the ball of the same radius containing the origin.

**Lemma 3.7.** *Let  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  be a coloring,  $V \subseteq k^n$  a sub-space and  $\pi: B \rightarrow K^d$  an exhibition of  $V$ . Then  $\chi$  is  $V$ -translatable if and only if there exists a definable family of risometries  $\alpha_x: B \rightarrow B$ , where  $x$  runs over  $\pi(B - B)$ , with the following properties (for all  $x, x' \in \pi(B)$  and all  $z \in \pi^{-1}(x)$ ):*

- (1)  $\chi \circ \alpha_x = \chi$
- (2)  $\alpha_x \circ \alpha_{x'} = \alpha_{x+x'}$
- (3)  $\pi(\alpha_x(z) - z) = x$
- (4)  $\text{dir}(\alpha_x(z) - z) \in V$

*Proof.* “ $\Rightarrow$ ”: Choose a straightener  $\phi$  respecting the fibers of  $\pi$  (using Lemma 3.6 (1)) and let  $\tilde{V}$  be the corresponding lift of  $V$ . For any  $x \in \pi(B - B)$ , denote by  $\alpha'_x: B \rightarrow B$  the translation by the unique element of  $\pi^{-1}(x) \cap \tilde{V}$ . Then  $\alpha'_x$  satisfies  $\chi \circ \phi \circ \alpha'_x = \chi \circ \phi$  and (2) – (4), and from this, one deduces that  $\alpha_x := \phi \circ \alpha'_x \circ \phi^{-1}$  satisfies (1) – (4).

“ $\Leftarrow$ ”: Without loss,  $0 \in B$  and  $V = k^d \times \{0\}^{n-d}$ ; write elements of  $B$  as  $(x, y) \in K^d \times K^{n-d}$ , and let  $\pi$  be the projection to the first  $d$  coordinates. We claim that  $\phi(x, y) := \alpha_x(0, y)$  is a straightener.

By (1),  $\chi \circ \phi$  is  $(K^d \times \{0\}^{n-d})$ -translation invariant. To check that  $\phi$  is a risometry, consider  $(x_1, y_1), (x_2, y_2) \in B$  and set  $x := x_2 - x_1$ . We have  $\phi(x_1, y_1) = \alpha_{x_1}(0, y_1)$  and (2) implies  $\phi(x_2, y_2) = \alpha_{x_1}(\alpha_x(0, y_2))$ , so since  $\alpha_{x_1}$  is a risometry, it suffices to check that  $\text{rv}((x_1, y_1) - (x_2, y_2)) = \text{rv}((0, y_1) - \alpha_x(0, y_2))$ ; but this follows  $v(y_2 - \pi^\vee(\alpha_x(0, y_2))) > v(x)$ , which in turn follows from (3) and (4).  $\square$

**Definition 3.8.** Let  $B \subseteq K^n$  be a ball,  $\pi: B \rightarrow K^d$  a coordinate projection, and  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  a coloring. A definable family of risometries  $(\alpha_x)_{x \in \pi(B - B)}$  from  $B$  to itself satisfying (1) – (4) of Lemma 3.7 will be called a *translater* of  $\chi$  (on  $B$ , with respect to  $\pi$ ).

Characterizing translatability via translators has the disadvantage of being more technical, but one advantage is that it avoids the (uncanonical) lift  $\tilde{V}$  appearing in the previous definition.

The following lemma says how translatability of a coloring is preserved under restriction to affine subspaces. For this to work, a transversality condition is needed.

**Lemma 3.9.** Suppose that  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  is a  $V$ -translatable coloring (where  $B \subseteq K^n$  is a ball and  $V \subseteq k^n$ ) and  $\rho: B \rightarrow K^d$  is a coordinate projection with  $\bar{\rho}(V) = k^d$  (in particular  $\dim V \geq d$ ). Then the restriction of  $\chi$  to any fiber  $\rho^{-1}(y)$  (for  $y \in \rho(B)$ ) is  $(V \cap \ker \bar{\rho})$ -translatable.

*Proof.* Choose an exhibition  $\pi: B \rightarrow K^d$  of  $V$  satisfying  $\ker \pi \subseteq \ker \rho$  and let  $\phi: B \rightarrow B$  be a straightener of  $\chi$  respecting the fibers of  $\pi$ . Then  $\phi$  sends any  $\rho$ -fiber  $\rho^{-1}(y)$  to itself and thus  $\phi|_{\rho^{-1}(y)}$  is a straightener of  $\chi|_{\rho^{-1}(y)}$  proving  $(V \cap \ker \bar{\rho})$ -translatability.  $\square$

The next two lemmas state that translatability behaves as one would expect with respect to dimension and topological closure (using the valued field topology); we write  $\overline{X}^{\text{top}}$  for the topological closure of a set  $X$ . Recall that we consider subsets of  $K^n$  as colorings, so that the definition of translatability can be applied to sets.

**Lemma 3.10.** Suppose that  $B \subseteq K^n$  is a ball, that  $X \subseteq B$  is a definable set which is  $V$ -translatable on  $B$  for some  $V \subseteq k^n$ , and that  $\pi: B \rightarrow K^d$  exhibits  $V$ . Then for any  $x \in \pi(B)$ , we have

$$\dim X = \dim(X \cap \pi^{-1}(x)) + d.$$

*Proof.* The translators of Lemma 3.7 can be restricted to definable bijections between the fibers  $X \cap \pi^{-1}(x)$ , so all of them have the same dimension. Now use Lemma 2.14 (2).  $\square$

**Lemma 3.11.** If  $X \subseteq K^n$  is  $V$ -translatable on a ball  $B \subseteq K^n$ , then so is  $(X, \overline{X}^{\text{top}})$ .

*Proof.* Since risometries are homomorphisms, a straightener for  $X$  also straightens  $\overline{X}^{\text{top}}$ .  $\square$

**3.2. Definition of t-stratifications.** We now give the general definition of t-stratification and prove basic properties. (The “t” in “t-stratification” stands for “translatable”.) Recall Convention 2.23 on how to treat a definable partition  $(S_i)_i$  of  $B_0$  as a coloring.

**Definition 3.12.** Let  $B_0 \subseteq K^n$  be a ball. A *t-stratification* of  $B_0$  is a partition of  $B_0$  into definable sets  $S_0, \dots, S_n$  with the properties listed below. We write  $S_{\leq d}$  for  $S_0 \cup \dots \cup S_d$  and  $S_{\geq d}$  for  $S_d \cup \dots \cup S_n$ .

- (1)  $\dim S_d \leq d$
- (2) For each  $d$  and each ball  $B \subseteq S_{\geq d}$  (open or closed),  $(S_i)_{i \leq n}$  is  $d$ -translatable on  $B$ .

We say that a t-stratification  $(S_i)_{i \leq n}$  *reflects* a coloring  $\chi: B_0 \rightarrow \text{RV}^{\text{eq}}$  if the following stronger version of (2) holds:

- (2') For each  $d$  and each ball  $B \subseteq S_{\geq d}$  (open or closed),  $((S_i)_{i \leq n}, \chi)$  is  $d$ -translatable on  $B$ .

Let us give a name to Condition (2').

**Definition 3.13.** For a definable partition  $(S_i)_i$  of  $B_0 \subseteq K^n$  and a coloring  $\chi: B_0 \rightarrow K^n$ , we say that  $((S_i)_i, \chi)$  is *sufficiently translatable* on a ball  $B \subseteq B_0$  if it is  $d$ -translatable, where  $d$  is the maximal integer such that  $B \subseteq S_{\geq d}$ .

Note that assuming  $\dim S_i \leq i$  for each  $i$ , sufficiently translatable also means “as translatable as possible”: by Lemma 3.10,  $S_d$  is at most  $d$ -translatable on  $B$  (this uses  $B \cap S_d \neq \emptyset$ ). In particular, it implies  $\text{tsp}_B(S_d) = \text{tsp}_B((S_i)_i) = \text{tsp}_B((S_i)_i, \chi)$ .

**Remark 3.14.** If  $(S_i)_i$  is a t-stratification of  $B_0$  (reflecting  $\chi$ ), then the restriction to any subball of  $B_0$  is also a t-stratification (reflecting the restriction of  $\chi$ ). In the other direction, a t-stratification of  $B_0 \subseteq K^n$  can be extended to a t-stratification of  $K^n$  by replacing  $S_n$  with  $S_n \cup (K^n \setminus B_0)$ , but only under the assumption that  $S_0 \neq \emptyset$ . This assumption is needed because in general,  $(S_i)_i$  will not be translatable on any ball strictly bigger than  $B_0$ .

In general, whether a coloring  $\chi$  is  $V$ -translatable on a ball  $B$  is not a definable property. However, for t-stratifications, it is:

**Lemma 3.15.** *Let formulas be given which define, in every model  $K \models \mathcal{T}$ , a t-stratification  $(S_i)_i$  of  $K^n$ , a ball  $B \subseteq K^n$  and a sub-vector space  $V \subseteq k^n$ . Then there exists a sentence  $\eta$  such that  $K \models \eta$  iff  $(S_i)_i$  is  $V$ -translatable on  $B$ .*

*Proof.* Let  $d$  be minimal such that  $B \cap S_d \neq \emptyset$  and let  $\pi: B \rightarrow K^d$  be an exhibition of  $W := \text{tsp}_B((S_i)_i)$ ; we have to find a first order way to describe  $W$ .

For each  $\pi$ -fiber  $F = \pi^{-1}(y)$  (with  $y \in \pi(B)$ ),  $S_d \cap F$  is finite by Lemma 3.10, so for any  $x \in S_d \cap F$ , we can find a ball  $B' \subseteq B$  such that  $B' \cap S_d \cap F = \{x\}$ . By  $W$ -translatability on  $B'$ ,  $B' \cap S_d \cap \pi^{-1}(y')$  is a singleton for any  $y' \in \pi(B')$  and we obtain  $W = \{\text{dir}(x_1 - x_2) \mid x_1, x_2 \in S_d \cap B'\}$ . Thus we have the following first order description:  $(S_i)_i$  is  $V$ -translatable on  $B$  iff for any  $x \in S_d$  (with  $d$  as above) and any sufficiently small ball  $B'$  containing  $x$ ,  $V \subseteq \{\text{dir}(x_1 - x_2) \mid x_1, x_2 \in S_d \cap B'\}$ .  $\square$

A property of t-stratifications which is important for inductive arguments is that on an affine subspace of a ball which is transversal to the translatability space on that ball, they again induce t-stratifications. This is the statement of the following lemma. (It is formulated for a t-stratification reflecting a coloring, but of course, we can apply it to the trivial coloring if we are interested in a “pure” t-stratification.)

**Lemma 3.16.** *Let  $(S_i)_i$  be a  $t$ -stratification of  $B_0 \subseteq K^n$  reflecting a coloring  $\chi: B_0 \rightarrow \text{RV}^{\text{eq}}$ , and let  $B \subseteq B_0$  be a ball. Let  $\pi: B \rightarrow K^d$  be an exhibition of  $\text{tsp}_B((S_i)_i)$  and suppose that  $F = \pi^{-1}(x)$  is a  $\pi$ -fiber (for some  $x \in \pi(B)$ ). Set  $T_i := S_{i+d} \cap F$  for  $0 \leq i \leq n-d$ ; then  $(T_i)_{i \leq n-d}$  is a  $t$ -stratification of  $F \cap B$  reflecting  $\chi|_{F \cap B}$ .*

*Proof.* By Lemma 3.10,  $\dim S_{i+d} \leq i+d$  implies  $\dim T_i \leq i$ , so it remains to show sufficient translatability. Consider a ball  $B' \subseteq B$  with  $B' \cap F \neq \emptyset$  and suppose that  $j$  is minimal with  $B' \cap F \subseteq T_{\geq j}$ . We have to show  $j$ -translatability on  $B' \cap F$ .

Set  $V := \text{tsp}_B((S_i)_i)$  and  $V' := \text{tsp}_{B'}((S_i)_i)$ . By  $V$ -translatability of  $S_{\geq j+d}$ ,  $B' \cap F \subseteq S_{\geq j+d}$  implies  $B' \subseteq S_{\geq j+d}$ , so  $\dim V' = j+d$ . Since  $V \subseteq V'$ , we have  $\overline{\pi}(V') = k^d$ , so Lemma 3.9 implies  $(V' \cap \ker \overline{\pi})$ -translatability of  $((S_i)_i, \chi)$  on  $B' \cap F$ . Now we are done since  $\dim(V' \cap \ker \overline{\pi}) = j$ .  $\square$

Here are some “global” properties of  $t$ -stratifications.

**Lemma 3.17.** *Let  $(S_i)_i$  be a  $t$ -stratification of  $B_0 \subseteq K^n$ . Then the following holds:*

- (1) *For each  $d$  and each  $x \in S_{\geq d+1}$ , there exists a maximal ball  $B$  containing  $x$  such that  $B \cap S_{\leq d} = \emptyset$ . Moreover, if  $B \neq B_0$  then  $B$  is open. In particular, the sets  $S_{\leq d}$  are topologically closed.*
- (2)  *$S_d$  has dimension exactly  $d$  locally at each point  $x \in S_d$ . In particular, either  $\dim S_d = d$  or  $S_d = \emptyset$ .*

*Proof.* (1) For  $d = 0$ , this is clear since  $S_0$  is finite; now suppose  $d > 0$ . By induction, there is a maximal ball  $B$  containing  $x$  with  $B \cap S_{\leq d-1} = \emptyset$ . If  $B \cap S_d = \emptyset$ , then  $B$  is the ball we are looking for, so suppose now that  $B \cap S_d \neq \emptyset$ . Then  $V := \text{tsp}_B((S_i)_i)$  is  $d$ -dimensional; let  $\pi: B \rightarrow K^d$  be an exhibition of  $V$  and let  $F \subseteq B$  be the  $\pi$ -fiber containing  $x$ . Since  $F \cap S_d$  is finite and non-empty, we find a maximal open ball  $B' \subseteq B$  such that  $B' \cap F \cap S_d = \emptyset$ . Now  $V$ -translatability implies  $B' \cap S_d = \emptyset$ , so  $B'$  is the ball we were looking for.

(2) Let  $x \in S_d$  be given. By (1), there exists a ball  $B$  containing  $x$  with  $B \subseteq S_{\geq d}$ , hence on any sub-ball  $B' \subseteq B$ , we have  $d$ -translatability. Now  $\dim(S_d \cap B') < d$  would contradict Lemma 3.10.  $\square$

**3.3. Characterizations of reflection.** A stratification is supposed to describe a coloring up to risometry. In this subsection, we will make this precise by giving different characterizations of when a stratification reflects a coloring. On our way, we will describe the finest coloring reflected by a given  $t$ -stratification.

**Definition 3.18.** Let  $(S_i)_i$  be a  $t$ -stratification of a ball  $B_0 \subseteq K^n$ . We define a *rainbow* of  $(S_i)_i$  to be a coloring  $\rho: B_0 \rightarrow \text{RV}^{\text{eq}}$  which is obtained by coloring a point  $x \in B_0$  with a code for the tuple of sets  $(\text{rv}(x - S_i))_{i \leq n}$ . (Recall that such a code exists in  $\text{RV}^{\text{eq}}$  by stable embeddedness of  $\text{RV}^{\text{eq}}$ .)

**Remark 3.19.** The rainbow  $\rho$  of  $(S_i)_i$  is a refinement of  $(S_i)_i$  viewed as a coloring, so any risometry  $B_0 \rightarrow B_0$  respecting  $\rho$  obviously respects  $(S_i)_i$  (i.e., sends each  $S_i$  to itself). Vice versa, if  $\phi$  is a risometry respecting  $(S_i)_i$ , then for any  $x \in B_0$  we have  $\text{rv}(x - S_i) = \text{rv}(\phi(x) - S_i)$  and hence  $\phi$  respects the rainbow.

**Proposition 3.20.** *Let  $(S_i)_i$  be a  $t$ -stratification of  $B_0 \subseteq K^n$  and let  $\chi: B_0 \rightarrow \text{RV}^{\text{eq}}$  be a coloring. Then the following are equivalent.*

- (1)  *$(S_i)_i$  reflects  $\chi$ .*

- (2) *The rainbow of  $(S_i)_i$  is a refinement of  $\chi$ .*  
 (3) *Any definable risometry  $\phi: B_0 \rightarrow B_0$  respecting  $(S_i)_i$  also respects  $\chi$ .*

*Proof.* (2)  $\Rightarrow$  (3) follows from Remark 3.19.

(3)  $\Rightarrow$  (1): For any ball  $B \subseteq B_0$ , we have to show that  $\text{tsp}_B((S_i)_i, \chi) = \text{tsp}_B((S_i)_i)$ . Let  $(\alpha_x)_x$  be a translator for  $(S_i)_i$  on  $B$ , with respect to any exhibition of  $\text{tsp}_B((S_i)_i)$  (see Definition 3.8). Extending each  $\alpha_x$  by the identity on  $B_0 \setminus B$  yields risometries  $\alpha_x: B_0 \rightarrow B_0$  respecting  $(S_i)_i$ . By (3), these risometries also respect  $\chi$ , hence  $(\alpha_x)_x$  is also a translator for  $((S_i)_i, \chi)$ .

(1)  $\Rightarrow$  (2): Let  $\rho$  be a rainbow of  $(S_i)_i$  and suppose that for two points  $y_1, y_2 \in B_0$ , we have  $\rho(y_1) = \rho(y_2)$  but  $\chi(y_1) \neq \chi(y_2)$ . Let  $B := B(y_1, \geq v(y_1 - y_2))$  be the smallest ball containing  $y_1$  and  $y_2$  and set  $V := \text{tsp}_B((S_i)_i)$ . We may assume that  $y_1, y_2$  have been chosen such that  $d := \dim V$  is maximal.

Choose an exhibition  $\pi: B \rightarrow K^d$  of  $V$  and a corresponding translator  $(\alpha_x)_{x \in \pi(B-B)}$  of  $((S_i)_i, \chi)$ . Set  $x_j := \pi(y_j)$  and let  $F_j := \pi^{-1}(x_j)$  be the fiber containing  $y_j$ . Then for  $y'_1 := \alpha_{x_2-x_1}(y_1) \in F_2$ , we have  $\chi(y'_1) = \chi(y_1) \neq \chi(y_2)$ . Moreover, since  $\alpha_{x_2-x_1}$  respects  $(S_i)_i$ , it also respects its rainbow (by Remark 3.19), i.e.,  $\rho(y'_1) = \rho(y_1) = \rho(y_2)$ .

Now set  $B' := B(y'_1, \geq v(y'_1 - y_2)) \subseteq B$ . It remains to show that  $B' \cap S_d = \emptyset$  to get a contradiction to the maximality of  $d$ . The set  $T := S_d \cap F_2$  is finite but non-empty. For any  $y \in F_2$ , we have  $\text{rv}(y - T) = \text{rv}(y - S_d) \cap \text{rv}(F_2 - F_2)$  by  $V$ -translatability on  $B$ , thus  $\rho(y'_1) = \rho(y_2)$  implies  $\text{rv}(y'_1 - T) = \text{rv}(y_2 - T)$ . Now, Lemma 2.20 implies  $(B' \cap F_2) \cap T = \emptyset$ , which in turn implies  $B' \cap S_d = \emptyset$ .  $\square$

**Remark 3.21.** The equivalence (1)  $\iff$  (2) implies that for any two colorings  $\chi_1, \chi_2$ ,  $(S_i)_i$  reflects the product  $(\chi_1, \chi_2)$  if and only if it reflects  $\chi_1$  and  $\chi_2$  separately.

**3.4. Families of sets up to risometry.** Given a definable family of sets  $(X_q)_{q \in Q}$  (or colorings), whether two sets  $X_q, X_{q'}$  are definably risometric defines an equivalence relation on  $Q$ . This equivalence relation is in general not definable; the main result of this subsection is that it does become definable if we equip each  $X_q$  with a  $t$ -stratification. Moreover, for each equivalence class, we can find a definable family of risometries which are compatible under composition. Under an additional assumption, we can even get some more information about these risometries; this will be needed in the proof of the main theorem. The assumption is the following, very weak variant of translatability.

**Definition 3.22.** Suppose that  $B \subseteq K^n$  is a definable subset (usually a ball),  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  is a coloring, and  $V \subseteq k^n$  is a vector space exhibited by  $\pi: B \rightarrow K^d$ . We say that  $\chi$  is *point-translatable* on  $B$  in direction  $V$  with respect to  $\pi$  (or simply  *$V$ - $\pi$ -point-translatable*) if for any  $y \in B$  and any  $x' \in \pi(B)$ , there exists an  $y' \in \pi^{-1}(x')$  with  $\chi(y') = \chi(y)$  and  $\text{dir}(y - y') \in V$ .

Notice the similarity to Condition (4) in Lemma 3.7 (the definition of translator).

**Proposition 3.23.** *Suppose that  $Q$  is a  $\emptyset$ -definable set (in any sort),  $(S_i)_{i \leq n}$  is a  $\emptyset$ -definable partition of  $Q \times K^n$  and  $\chi: Q \times K^n \rightarrow \text{RV}^{\text{eq}}$  is a  $\emptyset$ -definable coloring. Write  $\pi$  for the projection  $Q \times K^n \twoheadrightarrow Q$ . For  $q \in Q$ , set  $S_{i,q} := S_i \cap \pi^{-1}(q)$  and  $\chi_q := \chi|_{\pi^{-1}(q)}$ . Then we have the following:*

- (1) *The set  $Q' \subseteq Q$  of those  $q$  for which  $(S_{i,q})_{i \leq n}$  is a  $t$ -stratification of  $\{q\} \times K^n$  reflecting  $\chi_q$  is  $\emptyset$ -definable.*

- (2) There exists a  $\emptyset$ -definable coloring  $\chi': Q' \rightarrow \text{RV}^{\text{eq}}$  such that for all  $q, q' \in Q'$ ,  $\chi'(q) = \chi'(q')$  if and only if there exists a definable risometry  $\phi: \{q\} \times K^n \rightarrow \{q'\} \times K^n$  respecting the coloring  $((S_i)_i, \chi)$ .
- (3) For each  $\chi'$ -monochromatic piece  $C \subseteq Q'$ , there exists a compatible  $\ulcorner C \urcorner$ -definable family  $\alpha_{q,q'}: ((S_{i,q})_i, \chi_q) \rightarrow ((S_{i,q'})_i, \chi_{q'})$  of risometries, where  $q, q'$  run through  $C$ . Compatible means:  $\alpha_{q',q''} \circ \alpha_{q,q'} = \alpha_{q,q''}$ .

Suppose now that  $Q \subseteq K^m$  and that  $V \subseteq k^{m+n}$  is exhibited by the projection  $\pi$  from above. Then we also have the following variant of (3):

- (3') If, for some  $\chi'$ -monochromatic piece  $C \subseteq Q'$ ,  $(S_i)_i$  is  $V$ - $\pi$ -point-translatable on  $C \times K^n$  and moreover  $S_0 \cap (C \times K^n) \neq \emptyset$ , then the family  $\alpha_{q,q'}$  can be chosen such that additionally,  $\text{dir}(\alpha_{q,q'}(x) - x) \in V$  for all  $q, q' \in Q$  and all  $x \in \{q\} \times K^n$ .

All of the above works uniformly for all models  $K$  of our theory  $\mathcal{T}$ , i.e., given formulas defining  $Q$ ,  $S_i$  and  $\chi$ , we can find formulas defining  $Q'$ ,  $\chi'$  and  $\alpha_{q,q'}$  not depending on  $K$ .

**Remark 3.24.** Taking  $Q := \{0, 1\}$ , in particular we obtain: if there exists a definable risometry between two  $\emptyset$ -definable  $t$ -stratifications, then there already exists a  $\emptyset$ -definable one.

Before we prove the proposition, let us consider the following corollary which shows how statement (3') can be used to deduce translatability.

**Corollary 3.25.** Suppose that  $B \subseteq K^n$  is a ball,  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  is a coloring,  $V \subseteq k^n$  is a sub-space exhibited by  $\pi: B \rightarrow K^d$ , and  $(S_i)_{0 \leq i \leq n-d}$  is a definable partition of  $B$  such that for each  $\pi$ -fiber  $F \subseteq B$ ,  $(S_i \cap F)_i$  is a  $t$ -stratification reflecting  $\chi|_F$ . Suppose moreover that  $S_0$  is non-empty, that for any two  $\pi$ -fibers  $F, F'$  there exists a definable risometry  $\phi: F \rightarrow F'$  respecting  $((S_i)_i, \chi)$ , and that  $(S_i)_i$  is  $V$ - $\pi$ -point translatable on  $B$ . Then  $((S_i)_i, \chi)$  is  $V$ -translatable on  $B$ .

*Proof.* Without loss,  $\pi$  is the projection to the first  $d$  coordinates; set  $Q := \pi(B)$ . We extend the domains of  $\pi$ ,  $\chi$ , and  $(S_i)_i$  from  $B$  to  $Q \times K^{n-d}$ : for  $\chi$ , we use a single new color outside of  $B$ , and for  $(S_i)_i$ , we simply enlarge  $S_{n-d}$  (and keep all  $S_i$  for  $i < n-d$ ). Then for each  $q \in Q$ ,  $(S_i \cap \pi^{-1}(q))_i$  is a  $t$ -stratification of  $\pi^{-1}(q)$  reflecting  $\chi|_{\pi^{-1}(q)}$  (this uses  $S_0 \cap \pi^{-1}(q) \neq \emptyset$ ; cf. Remark 3.14). When applying Proposition 3.23 to this data, the whole set  $Q$  becomes a single  $\chi'$ -monochromatic piece which satisfies the prerequisites of (3'), hence we obtain a single family of risometries  $\alpha_{q_1, q_2}: \pi^{-1}(q_1) \rightarrow \pi^{-1}(q_2)$  as in (3'). Define a family  $(\beta_q)_{q \in Q-Q}$  of maps  $B \rightarrow B$  by  $\beta_q(x) := \alpha_{\pi(x), \pi(x)+q}(x)$ . We claim that this family is a translator proving  $V$ -translatability of  $((S_i)_i, \chi)$  on  $B$ .

It is clear that these  $\beta_q$  satisfy Conditions (1) – (4) of Lemma 3.7 (by definition of  $\beta_q$  and by the properties of  $\alpha_{q_1, q_2}$ ), so it remains to check that each  $\beta_q$  is a risometry. To see this, choose  $x_1, x_2 \in B$  and set  $q_i := \pi(x_i)$  and  $x_3 := \alpha_{q_2, q_1}(x_2)$ . Then  $\beta_q$  preserves both,  $\text{rv}(x_1 - x_3)$  (since  $\alpha_{q_1, q_1+q}$  is a risometry) and  $\text{rv}(x_3 - x_2)$  (since  $\pi(x_3 - x_2)$  and  $\text{dir}(x_3 - x_2)$  are preserved), and these two values together determine  $\text{rv}(x_1 - x_2)$  by Lemma 2.6 (2).  $\square$

*Proof of Proposition 3.23.* The whole proof is by induction on  $n$ , i.e. we assume that the proposition holds for smaller  $n$ .

(1) Here is an informal formula defining  $Q'$ :

$$\begin{aligned}
& \bigwedge_{d=0}^n \dim S_{d,q} \leq d \\
& \wedge \bigwedge_{d=1}^n \forall \text{ balls } B \subseteq S_{\geq d,q} \text{ with } B \cap S_{d,q} \neq \emptyset : \\
& \quad \bigvee_{\substack{\rho: B \rightarrow \{q\} \times K^d \\ \text{coordinate} \\ \text{projection}}} \exists V \subseteq k^n \text{ sub-space} : \\
& \quad \quad \rho \text{ exhibits } V \\
& \quad \quad \wedge (S_{i,q})_i \text{ is } V\text{-}\rho\text{-point-translatable on } B \\
& \quad [For } x \in \rho(B), \text{ set } T_{i,x} := S_{i+d,q} \cap \rho^{-1}(x) \text{ and } \chi_x := \chi_q|_{\rho^{-1}(x)}] \\
& \quad \quad \wedge (T_{i,x})_{i \leq n-d} \text{ is a t-stratification reflecting } \chi_x \text{ for all } x \in \rho(B) \\
& \quad \quad \wedge \text{ all } ((T_{i,x})_i, \chi_x) \text{ are definably risometric for } x \in \rho(B)
\end{aligned}$$

This is first order: in the first line, we use that dimension is definable; in the last two lines, we use (1), (2) of the induction hypothesis.

If  $(S_{i,q})_i$  is a t-stratification reflecting  $\chi_q$ , then it is clear the formula holds. For the other direction, note that by the induction hypothesis and Corollary 3.25, the last four lines of the formula together with  $B \cap S_{d,q} \neq \emptyset$  imply that  $((S_{i,q})_i, \chi_q)$  is  $V$ -translatable on  $B$ .

**(2) and (3)** Without loss,  $Q = Q'$ . Moreover, if we have a coloring  $\chi': Q \rightarrow \text{RV}^{\text{eq}}$  such that the existence of a risometry  $\{q\} \times K^n \rightarrow \{q'\} \times K^n$  respecting  $((S_i)_i, \chi)$  implies  $\chi'(q) = \chi'(q')$ , we can consider each  $\chi$ -monochromatic piece separately for the remainder of the proof (adding the color of the piece to the language). We will do this several times; at the end, we will obtain a definable compatible family of risometries on the whole of  $Q$ , thus proving both (2) and (3).

By Lemma 2.20 (1), there is at most one risometry sending  $S_{0,q}$  to  $S_{0,q'}$ . Whether such a risometry exists can definably be tested by choosing an enumeration  $(x_\mu)_\mu$  of  $S_{0,q}$  and comparing the matrix  $(\text{rv}(x_\mu - x_\nu))_{\mu,\nu}$  to a corresponding matrix for  $S_{0,q'}$ . (Note that the cardinality  $|S_{0,q}|$  is bounded.) Thus we can suppose that for each  $q, q' \in Q$ , a risometry  $\beta_{q,q'}: S_{0,q} \rightarrow S_{0,q'}$  exists and that  $\beta_{q,q'}$  respects  $\chi|_{S_0}$ . Moreover (again by uniqueness of this risometry),  $\beta_{q,q'}$  is  $(q, q')$ -definable and the family  $(\beta_{q,q'})_{q,q'}$  is compatible with composition (as required in (3)).

Consider a set  $R \subseteq \text{RV}^{(n)}$  such that  $B_{R,q} := \{x \in \{q\} \times K^n \mid \text{rv}(x - S_{0,q}) = R\}$  is non-empty. By Lemma 2.20, this non-emptiness condition does not depend on  $q$ ,  $B_{R,q}$  is a maximal ball not intersecting  $S_{0,q}$  (possibly equal to  $\{q\} \times K^n$ ), and any risometry  $\{q\} \times K^n \rightarrow \{q'\} \times K^n$  respecting  $S_0$  sends  $B_{R,q}$  to  $B_{R,q'}$ . This means that we can treat each family  $(B_{R,q})_q$  separately as follows. For each  $R$  as above, we will construct a coloring  $\chi'_R$  of  $Q$  and a definable compatible family of risometries  $\alpha_{R,q,q'}: B_{R,q} \rightarrow B_{R,q'}$  such that (2) and (3) hold for the restricted ( $\ulcorner R \urcorner$ -definable) family  $((S_{i,q} \cap B_{R,q})_i, \chi_q|_{B_{R,q}})_{q \in Q}$ . By compactness (and using that  $\chi'_R$  and  $\alpha_{R,q,q'}$  are definable without additional parameters), we can assume that the definitions of  $\chi'_R$  and  $\alpha_{R,q,q'}$  are uniform in  $\ulcorner R \urcorner$ . Using stable embeddedness of  $\text{RV}^{\text{eq}}$  (Hypothesis 2.8 (1)), we define a “total” coloring  $\chi': q \mapsto \ulcorner \ulcorner R \urcorner \mapsto \chi'_R(q) \urcorner$ . For  $q, q' \in Q$  with  $\chi'(q) = \chi'(q')$ , the risometries  $\beta_{q,q'}$  and  $\alpha_{R,q,q'}$  can be assembled

to a definable map  $\alpha_{q,q'}: K^n \rightarrow K^n$ ; this map is a risometry by Lemma 2.20, and for varying  $q, q'$  we have the required compatibility.

Thus from now on fix  $R$ , add  $\lceil R \rceil$  to the language, and to simplify notation, we write  $B_q$  instead of  $B_{R,q}$ . Moreover, we set  $B := \bigcup_{q \in Q} B_q \subseteq Q \times K^n$ .

For some  $q \in Q$ , set  $W := \text{tsp}_{B_q}((S_{i,q})_i)$ . By Lemma 3.15,  $W$  is  $q$ -definable, we may color  $Q$  with  $\lceil W \rceil$ ; in other words, we may assume that  $W$  does not depend on  $q$ . Set  $d := \dim W$ , choose an exhibition  $\rho': K^n \twoheadrightarrow K^d$  of  $W$ , and set  $\rho: B \rightarrow Q \times K^d, (q, y) \mapsto (q, \rho'(y))$ . By Lemma 3.16, we get a family of  $t$ -stratifications of the fibers of  $\rho$ , parametrized by  $\tilde{Q} := \rho(B)$ . Applying induction (2) to this family (after using Remark 3.14) yields a coloring  $\tilde{\chi}$  of  $\tilde{Q}$  depending only on the  $Q$ -coordinate. By Lemma 3.6, if  $B_q$  and  $B_{q'}$  are risometric (colored with  $((S_i)_i, \chi)$ ), then so are all the  $\rho$ -fibers, so we can assume that  $\tilde{Q}$  is  $\tilde{\chi}$ -monochromatic. Now induction (3) yields a definable compatible family  $\tilde{\alpha}_{\tilde{q},\tilde{q}'}: B_{\tilde{q}} \rightarrow B_{\tilde{q}}$  of risometries between the  $\rho$ -fibers.

To finish the construction of a definable compatible family of risometries  $\alpha_{q,q'}: B_q \rightarrow B_{q'}$ , it remains to find a definable compatible family of risometries  $\gamma_{q,q'}: \rho(B_q) \rightarrow \rho(B_{q'})$  (which does not need to respect any coloring); after that, we can set  $\alpha_{q,q'}(y) := \tilde{\alpha}_{\tilde{q},\tilde{q}'}(y)$ , where  $\tilde{q} := \rho(y)$  and  $\tilde{q}' := \gamma_{q,q'}(\tilde{q})$ .

If  $S_0$  is empty, then  $B_q = \{q\} \times K^n$  and we can set  $\gamma_{q,q'}(q, z) := (q', z)$  for every  $q, q' \in Q, z \in K^n$ , so suppose now that  $S_0$  is non-empty. For each  $q$ , let  $N_q$  consist of those elements of  $S_{0,q}$  which are closest to  $B_q$  (or, equivalently,  $N_q = S_{0,q} \cap \bar{B}_q$ , where  $\bar{B}_q$  is the unique closed ball containing  $B_q$  with  $\text{rad}_c(\bar{B}_q) = \text{rad}_o(B_q)$ ). Define  $c_q := \frac{1}{|N_q|} \sum_{s \in N_q} s$  to be the barycentrum of  $N_q$ . The translation  $\tilde{\gamma}_{q,q'}: y \mapsto y - c_q + c_{q'}$  sends  $B_q$  to  $B_{q'}$ , so we can define  $\gamma_{q,q'}: \rho(B_q) \rightarrow \rho(B_{q'})$  to be the induced map on the projections.

**(3')** Let us say that a map  $\phi$  between two subsets of  $Q \times K^n$  “moves in direction  $V$ ” if  $\text{dir}(\phi(x) - x) \in V$  for all  $x$  in the domain. We claim that under the additional assumptions of (3'), the maps  $\alpha_{q,q'}$  constructed in the proof of (3) do already move in direction  $V$ ; so let us go through the construction of  $\alpha_{q,q'}$ .

First, we have to check that the risometries  $\beta_{q,q'}: S_{0,q} \rightarrow S_{0,q'}$  move in direction  $V$ . Set  $\delta := v(q - q')$ , and let us say that  $T \subseteq S_{0,q'}$  is a set of  $\delta$ -representatives (of  $S_{0,q'}$ ) if for each  $s \in S_{0,q'}$  there exists exactly one  $t \in T$  with  $v(s - t) > \delta$ . Choose any set of  $\delta$ -representatives  $T \subseteq S_{0,q'}$ . For each  $t \in T$ , using point-translatibility of  $S_0$ , we can choose an element  $\phi(t) \in S_{0,q}$  with  $\text{dir}(\phi(t) - t) \in V$ . Using that  $v(t - t') \leq v(q - q')$  for any two different  $t, t' \in T$ , we get that  $\phi: T \rightarrow \phi(T)$  is a risometry. Composing with  $\beta_{q,q'}$  yields a risometry from  $T$  to  $T' := \beta_{q,q'}(\phi(T))$ , and  $T'$  is also a set of  $\delta$ -representatives of  $S_{0,q'}$ . The bijection from  $T$  to  $T'$  sending  $t$  to the unique  $t' \in T'$  with  $v(t - t') > \delta$  is also a risometry, so by Lemma 2.20, it is equal to  $\beta_{q,q'} \circ \phi$ . This implies that  $\text{dir}(y - \beta_{q,q'}(y)) \in V$ , first for  $y \in \phi(T)$  and then also for all other  $y \in S_{0,q}$ .

Now, to get that the maps  $\alpha_{q,q'}: B_q \rightarrow B_{q'}$  move in direction  $V$ , it remains to check that both, the maps  $\tilde{\alpha}_{\tilde{q},\tilde{q}'}$  and the maps  $\tilde{\gamma}_{q,q'}$  move in direction  $V$ . Let us first consider the maps  $\tilde{\gamma}_{q,q'}$ . By assumption,  $S_0 \neq \emptyset$ , so  $\tilde{\gamma}_{q,q'}(y) = y - c_q + c_{q'}$ , which moves in direction  $V$  since  $\beta_{q,q'}(N_q) = N_{q'}$  and  $\beta_{q,q'}$  moves in direction  $V$ .

To obtain that the maps  $\tilde{\alpha}_{\tilde{q},\tilde{q}'}$  move in direction  $V$ , we will apply (3') instead of (3) in the induction. For this, we take  $\tilde{V} := V + (\{0\}^m \times W) \subseteq k^{m+n}$ . Since  $B_q \cap S_{d,q} \neq \emptyset$ , the 0-dimensional stratum of the induction is non-empty, and it remains to check point-translatibility: for given  $(q, x), (q', x') \in \tilde{Q}$  and  $y \in \rho^{-1}(q, x)$ , we

need to find an element  $y' \in \rho^{-1}(q', x')$  of the right color (i.e., in the same set  $S_i$  as  $y$ ) which satisfies  $\text{dir}(y - y') \in \tilde{V}$ .

Set  $\delta := v(q - q')$ . If  $\delta \leq \text{rad}_o B_q$ , then any  $y' \in B_{q'}$  satisfies  $\text{dir}(y - y') \in V$ , since  $\tilde{\gamma}_{q,q'}$  moves in direction  $V$  and sends  $B_q$  to  $B_{q'}$ , so the risometry  $B_q \rightarrow B_{q'}$  from the proof of (3) yields an  $y'$  with the desired properties.

If  $\delta > \text{rad}_o B_q$ , then let  $y'' \in \pi^{-1}(q')$  be a point of the same color as  $y$  obtained from the  $V$ -point-translatibility in the assumptions. Using again that  $\tilde{\gamma}_{q,q'}$  moves in direction  $V$ , we get  $v(y'' - \tilde{\gamma}_{q,q'}(y)) > \delta$  and thus  $y'' \in B_{q'}$ . Now we use  $W$ -translatibility of  $B_{q'}$  to move  $y''$  to the fiber  $\rho^{-1}(q', x')$ .  $\square$

A priori, being a  $t$ -stratification is not first order, since there might be no bound on how complicated the straighteners in a single  $t$ -stratification are. However, Proposition 3.23 (1) says that after all, being a  $t$ -stratification *is* first order; from this, we can deduce a posteriori that all straighteners appearing in a single  $t$ -stratification can be defined uniformly. (In fact, these uniformly defined straighteners can also directly be extracted from the proof of Proposition 3.23.)

**Corollary 3.26.** *If  $(S_i)_i$  is a  $t$ -stratification reflecting a coloring  $\chi: B_0 \rightarrow \text{RV}^{\text{eq}}$ , then the straighteners on all balls can be defined uniformly, i.e., there is a formula  $\eta(x, x', y)$ , where  $x, x'$  are  $n$ -tuples of valued field variables and  $y$  is an arbitrary tuple of variables, such that for any ball  $B \subseteq S_{\geq d}$ , there exists an element  $b$  such that  $\eta(x, x', b)$  defines the graph of a straightener of  $((S_i)_i, \chi)$  on  $B$  proving  $d$ -translatibility.*

*Proof.* For any formula  $\eta(x, x', y)$ , let  $\text{str}_\eta(b, \ulcorner B \urcorner)$  be a formula expressing that  $\eta(x, x', b)$  defines a straightener which proves sufficient translatibility of  $((S_i)_i, \chi)$  on the ball  $B$ . Applying Proposition 3.23 (1) to  $((S_i)_i, \chi)$  (where  $Q$  is a one-point-set) yields a sentence  $\psi$  which holds in  $K$  and such that for any model  $K' \models (\mathcal{T}, \psi)$  and any ball  $B \subseteq (K')^n$ , a straightener exists, i.e., there exists an  $\eta(x, x', y)$  (depending on  $K'$  and  $B$ ) such that  $K' \models \exists b \text{str}_\eta(b, \ulcorner B \urcorner)$ . By compactness, there is a single  $\eta(x, x', y)$  such that  $\mathcal{T} \cup \{\psi\}$  implies  $\forall B \exists b \text{str}_\eta(b, \ulcorner B \urcorner)$ ; in particular,  $\eta$  defines all straighteners in  $K$ .  $\square$

#### 4. PROOF OF EXISTENCE OF $T$ -STRATIFICATIONS

We now come to the proof of the main theorem about existence of  $t$ -stratifications, i.e., Theorem 4.10. Here is a very rough sketch of the proof (omitting many technicalities). Suppose we have a coloring  $\chi$  of  $K^n$ . The overall idea is to construct the sets  $S_d$  one after the other, starting with  $S_n$ . Suppose that  $S_n, \dots, S_{d+1}$  are already constructed and let  $X := K^n \setminus S_{\geq d+1}$  be the remainder, which we suppose to be of dimension at most  $d$ . To obtain  $S_d$ , we only have to find a set  $X' \subseteq X$  which is at most  $(d-1)$ -dimensional such that on any ball not intersecting  $X'$ , we have (at least)  $d$ -translatibility; then we can set  $S_d := X \setminus X'$ . However, to be able to obtain such an  $X'$  in a definable way, we have to drop the condition  $X' \subseteq X$ ; this is not a problem: we simply shrink the sets  $S_i$  we already constructed before (removing  $X'$  from them).

To prove  $d$ -translatibility on many balls  $B$ , we roughly proceed as follows. First, we use Lemma 4.4 to refine our coloring in such a way that each monochromatic piece  $C$  is  $(\dim C)$ -translatable separately on suitable balls (more precisely, we obtain that each  $C$  is “sub-affine”; see Definition 4.1). Merging the individual translatabilities of the monochromatic pieces  $C$  into  $d$ -translatibility of the whole

coloring (on a given ball  $B$ ) is done in Lemma 4.6. A main ingredient to this is that for a suitable coordinate projection  $\pi: B \rightarrow K^d$ , any two  $\pi$ -fibers are risometric. To obtain such risometries between fibers, we proceed in two different ways, depending on whether  $d = 1$  or  $d \geq 2$ .

Note that the fibers we are interested in are  $(n - d)$ -dimensional. In the case  $d \geq 2$ , we can apply induction to  $(n - d + 1)$ -dimensional affine subspaces. This yields many risometries inside such a space, and using Lemma 4.7, these “many” will turn out to be “enough”: we obtain a  $(d - 1)$ -dimensional “bad” set  $X'$  (as above) such that for any ball  $B$  not intersecting  $X'$ , there exists a  $\pi: B \rightarrow K^d$  such that all  $\pi$ -fibers are risometric.

In the case  $d = 1$ , this method does not work, since we can apply induction at most to  $(n - 1)$ -dimensional subspaces, each of which contains only a single  $\pi$ -fiber (for a given  $\pi: K^n \rightarrow K$ ). However, we can apply Proposition 3.23 to the family of all  $\pi$ -fibers, which again implies that many fibers are risometric. This alone still would leave a bad set  $X'$  which is far too big, but by doing this for all projections  $\pi: K^n \rightarrow K$  and moreover repeating it several times, we finally obtain a finite set  $X'$ , which we then can take as  $S_0$ . (This approach for  $d = 1$  would not work for higher  $d$  since we explicitly use that  $\pi$  goes to  $K$  itself and not to a Cartesian power of  $K$ .)

**4.1. Sub-affine pieces.** To get translatability of a coloring  $\chi: K^n \rightarrow \text{RV}^{\text{eq}}$  on certain balls, the first step is to refine it such that each monochromatic piece  $C$  is a subset of an “affine space up to smaller terms” of the same dimension as  $C$ . The following definition makes this precise.

**Definition 4.1.** Suppose  $C \subseteq K^n$  is a subset. We define the *affine direction space* of  $C$  to be the sub-space  $\text{affdir}(C) \subseteq k^n$  generated by  $\text{dir}(x - x')$ , where  $x, x'$  run through  $C$  (and  $x \neq x'$ ). We call  $C$  *sub-affine* (in direction  $\text{affdir}(C)$ ) if for every  $x \in C$ ,  $\dim_x(C) = \dim(\text{affdir}(C))$ .

Note that the intersection of a sub-affine set with a ball is again sub-affine. Also, one easily checks that for any definable  $C \subseteq K^n$ ,  $\dim(\text{affdir}(C)) \geq \dim C$ : if  $\pi$  exhibits  $\text{affdir}(C)$ , then each  $\pi$ -fiber contains at most one point of  $C$ . Thus “sub-affine” in particular means that  $\text{affdir}(C)$  is as small as possible.

Being sub-affine is closely related to translatability:

**Lemma 4.2.** *Let  $B \subseteq K^n$  be a ball and  $C \subseteq B$  a definable subset.*

- (1) *If  $C$  is  $V$ -translatable on  $B$  for some  $V \subseteq k^n$ , then  $V \subseteq \text{affdir}(C)$ .*
- (2) *If there is an exhibition  $\pi: B \rightarrow K^d$  of  $V := \text{affdir}(C)$  with  $\pi(C) = \pi(B)$ , then  $C$  is  $V$ -translatable on  $B$ .*

*Proof.* (1) Clear.

(2) Without loss,  $\pi$  is the projection to the first  $d$  coordinates; write elements of  $B$  as  $(x, y) \in K^d \times K^{n-d}$ . For any  $x \in \pi(B)$ , the fiber  $\pi^{-1}(x) \cap C$  consists of a single element  $(x, c(x))$ : it is non-empty by assumption, and two different elements  $(x, y), (x, y')$  would violate  $\text{dir}((x, y) - (x, y')) \in V$ . Assuming without loss  $0 \in B$ , we obtain a straightener  $\phi: B \rightarrow B$  by setting  $\phi(x, y) := \phi(x, y + c(x))$ .  $\square$

Now comes the place where we really need the Jacobian property. The next lemma can be seen as a multi-dimensional version of it. We denote the standard scalar product on  $K^n$  by  $\langle \cdot, \cdot \rangle$ . Moreover, in the remainder of this subsection, we

will use the convention  $\lambda + \infty = \infty$  for  $\lambda \in \Gamma$  and we will use the following notation. For  $\lambda_1, \lambda_2 \in \Gamma$ , set

$$\lambda_1 \dot{<} \lambda_2 : \Longleftrightarrow \lambda_1 < \lambda_2 \vee \lambda_1 = \lambda_2 = \infty.$$

(The usefulness of this comes from the fact that for any  $x, x' \in K^n$ , we have  $\text{rv}(x) = \text{rv}(x')$  iff  $v(x - x') \dot{>} v(x)$ .)

**Lemma 4.3.** *Let  $f: K^n \rightarrow K$  be a definable map. Then there exists a coloring  $\chi: K^n \rightarrow \text{RV}^{\text{eq}}$  such that for each monochromatic piece  $C = \chi^{-1}(\sigma)$  (with  $\sigma \in \text{im } \chi$ ), there exists an element  $\xi \in \text{RV}^{(n)}$  such that for any  $x, x' \in C, x \neq x'$ , we have*

$$“v(f(x) - f(x') - \langle \xi, x - x' \rangle) \dot{>} v_{\text{RV}}(\xi) + v(x - x')”;$$

more precisely, for any  $z \in K^n$  with  $\text{rv}(z) = \xi$ , we have

$$(*) \quad v(f(x) - f(x') - \langle z, x - x' \rangle) \dot{>} v(z) + v(x - x').$$

*Proof.* By fixing all but one coordinate, we consider  $f$  as a function in one variable (the other coordinates being parameters) and apply the Jacobian property (Definition 2.9). Doing this for each coordinate yields a coloring  $\chi: K^n \rightarrow \text{RV}^{\text{eq}}$  such that for each monochromatic piece  $C$  and for each  $x = (x_1, \dots, x_n), x' = (x'_1, \dots, x'_n) \in C, x \neq x'$ , we have the following. If  $x$  and  $x'$  differ only in the  $i$ -th coordinate, then  $\text{rv}(\frac{f(x) - f(x')}{x_i - x'_i}) = \xi_i$ , where  $\xi_i \in \text{RV}$  only depends on  $C$  and  $i$  (and not on  $x, x'$ ). By choosing any  $z'_i \in \text{rv}^{-1}(\xi_i)$ , this can be reformulated as

$$(**) \quad v(f(x) - f(x') - z'_i \cdot (x_i - x'_i)) \dot{>} v(z'_i) + v(x_i - x'_i).$$

Let  $\xi \in \text{RV}^{(n)}$  be the image of  $(\xi_1, \dots, \xi_n) \in \text{RV}^n$  under the canonical map and let  $z = (z_1, \dots, z_n) \in K^n$  be a preimage of  $\xi$ . Then  $v(z_i - z'_i) \dot{>} v_{\text{RV}}(\xi)$  and hence (\*\*) implies (for  $x, x'$  still differing only in the  $i$ -th coordinate)

$$(***) \quad v(f(x) - f(x') - z_i \cdot (x_i - x'_i)) \dot{>} v_{\text{RV}}(\xi) + v(x_i - x'_i).$$

Now we obtain (\*) for arbitrary  $x, x' \in C, x \neq x'$  by applying (\*\*\*) repeatedly, changing the coordinates one after the other.  $\square$

**Lemma 4.4.** *Let  $\chi: K^n \rightarrow \text{RV}^{\text{eq}}$  be a  $\emptyset$ -definable coloring. Then there exists a  $\emptyset$ -definable refinement  $\chi'$  of  $\chi$  such that each monochromatic piece  $C' \subseteq K^n$  of  $\chi'$  is sub-affine.*

*Proof.* We will prove the following claim: For any  $\emptyset$ -definable set  $C \subseteq K^n$  of dimension  $d$ , there is a  $\emptyset$ -definable coloring  $\tilde{\chi}$  of  $C$  such that each monochromatic piece  $\tilde{C} \subseteq C$  of dimension  $d$  satisfies  $\dim(\text{affdir}(\tilde{C})) = d$ .

Once we have this, we can finish the proof of the lemma as follows. We do an induction over the maximum of the dimensions of monochromatic pieces of  $\chi$  which are not sub-affine. Denote this maximum by  $d$ . On each  $\chi$ -monochromatic piece  $C$  of dimension  $d$  which is not sub-affine, we refine  $\chi$  as follows.

First, we apply the claim to  $C$  (with  $\ulcorner C \urcorner$  added to the language), which yields a  $\ulcorner C \urcorner$ -definable coloring  $\tilde{\chi}$  of  $C$ . Now consider a  $\tilde{\chi}$ -monochromatic piece  $\tilde{C}$  of dimension  $d$ . By Lemma 2.15, the set  $\tilde{D} := \{x \in \tilde{C} \mid \dim_x \tilde{C} < d\}$  has dimension less than  $d$ , so  $\tilde{C} \setminus \tilde{D}$  is sub-affine. Refine  $\tilde{\chi}$  such that each  $\tilde{C} \setminus \tilde{D}$  becomes a separate monochromatic piece. The result is our desired refinement of  $\chi$ . By construction, each monochromatic piece of the refinement of dimension  $d$  is sub-affine, so the induction works.

To prove the claim, we construct the coloring  $\tilde{\chi}$  of  $C$  as the product of one coloring  $\chi_\pi$  for each coordinate projection  $\pi: K^n \rightarrow K^d$ . Define  $\chi_\pi$  as follows:

- (1) All  $x \in C$  which are non-isolated in their fiber  $\pi^{-1}(\pi(x)) \cap C$  are painted in black; let  $C_1$  be the remainder; note that  $C_1$  has only finitely many points in each  $\pi$ -fiber.
- (2) By b-minimality, we get a  $\emptyset$ -definable map  $\rho: C_1 \rightarrow \text{RV}^{\text{eq}}$  which is injective on each  $\pi$ -fiber.
- (3) A  $\rho$ -monochromatic piece  $C' = \rho^{-1}(\sigma) \subseteq C_1$  can be seen as the graph of a  $\sigma$ -definable function  $f: \pi(C') \rightarrow K^{n-d}$ . Applying Lemma 4.3 to a coordinate  $\pi(C') \rightarrow K$  of  $f$  yields a coloring of  $\pi(C')$ ; let  $\rho'_\sigma$  be the product of those colorings for all coordinates. We use  $\rho'_\sigma$  to refine  $\rho$  on  $C'$ : for  $x \in C_1$ , set  $\chi_\pi(x) := (\rho(x), \rho'_{\rho(x)}(\pi(x)))$ .

It remains to check that if  $\tilde{C}$  is  $\tilde{\chi}$ -monochromatic and of dimension  $d$ , then  $\dim(\text{affdir}(\tilde{C})) = d$ , so assume for contradiction that  $d' := \dim(\text{affdir}(\tilde{C})) > d$ . Choose an exhibition  $\pi': K^n \rightarrow K^{d'}$  of  $V := \text{affdir}(\tilde{C})$ . Then  $\dim \pi'(\tilde{C}) = d$  since otherwise, there would be  $\pi'$ -fibers containing several points of  $\tilde{C}$ , contradicting that  $\pi'$  exhibits  $V$ . Next choose a coordinate projection  $\rho: K^{d'} \rightarrow K^d$  such that for  $\pi := \rho \circ \pi'$ , we still have  $\dim \pi(\tilde{C}) = d$ . Then  $\chi_\pi(\tilde{C})$  is not black, since otherwise,  $\dim(\pi^{-1}(x) \cap \tilde{C}) > 0$  for all  $x \in \pi(\tilde{C})$ .

To simplify notation, we now assume that  $\pi$  projects onto the first  $d$  coordinates and that the projection of  $V$  to the first  $d+1$  coordinates is still surjective; denote the latter projection by  $\pi'': K^n \rightarrow K^{d+1}$ , write  $\rho': K^{d+1} \rightarrow K^d$  for the projection to the first  $d$  coordinates, and write elements of  $K^{d+1}$  as  $(x, y)$  for  $x \in K^d, y \in K$ .

Let  $f: \pi(C') \rightarrow K$  be the map whose graph is  $\pi''(\tilde{C})$  (i.e.,  $f$  is the first coordinate of the function whose graph is  $\tilde{C}$ ), and let  $z \in K^d$  be as in Lemma 4.3 (\*) (applied to  $f$ ).

By definition of  $V$  and since  $\pi''(V) = k^{d+1}$ , we can find  $d+1$  pairs of points  $x'_i, x''_i \in \pi(\tilde{C})$  with  $x'_i \neq x''_i$  such that  $(\text{dir}((x'_i, f(x'_i)) - (x''_i, f(x''_i))))_i$  is a basis of  $k^{d+1}$ . Set  $x_i := x'_i - x''_i$  and  $y_i := f(x'_i) - f(x''_i)$ . Using this, the inequality of Lemma 4.3 becomes

$$(+) \quad v(y_i - \langle z, x_i \rangle) \stackrel{*}{>} v(z) + v(x_i).$$

Suppose first that  $v(z) < 0$ . Choose  $i$  with  $\langle \text{dir}(z), \overline{\rho}'(\text{dir}(x_i, y_i)) \rangle \neq 0$ . Then by Lemma 2.6 (3) we have  $\overline{\rho}'(\text{dir}(x_i, y_i)) = \text{dir}(x_i)$  and  $v(y_i) \geq v(x_i) > v(z) + v(x_i)$ ; moreover, Lemma 2.6 (4) implies  $v(\langle z, x_i \rangle) = v(z) + v(x_i)$ . Thus we have  $v(y_i - \langle z, x_i \rangle) = v(z) + v(x_i) < \infty$ , contradicting (+).

Now suppose  $v(z) \geq 0$ . Then (+) implies  $v((x_i, y_i) - (x_i, \langle z, x_i \rangle)) > v((x_i, y_i))$  and hence  $\text{dir}((x_i, y_i)) = \text{dir}((x_i, \langle z, x_i \rangle))$ . Using  $v(z) \geq 0$  again,  $\text{dir}(x)$  determines  $\text{dir}((x, \langle z, x \rangle))$ , i.e., we obtain a  $d$ -dimensional space  $\text{dir}(\{(x, \langle z, x \rangle) \mid x \in K^d\}) \subseteq k^{d+1}$  containing all  $\text{dir}(x_i, y_i)$ . This contradicts that  $\text{dir}(x_i, y_i)$  is a basis of  $k^{d+1}$ .  $\square$

**4.2. Merging translatability.** In the previous subsection, we obtained some first translatability separately for each monochromatic piece of a coloring. Now we will show how this can be merged to translatability of the whole coloring (under a lot of technical assumptions). We start with a lemma which allows us to relate affine direction spaces of different monochromatic pieces.

**Lemma 4.5.** *Let  $B \subseteq K^n$  be a ball and let  $C, C' \subseteq B$  be non-empty definable subsets which are sub-affine in directions  $V$  and  $V'$ , respectively. Suppose that*

$\pi: B \rightarrow K^d$  exhibits  $V$ , and finally suppose that any two fibers  $\pi^{-1}(y_1), \pi^{-1}(y_2)$  (for  $y_i \in \pi(B)$ ) are risometric, when colored with  $(C, C')$ . Then  $V \subseteq V'$ .

*Proof.* It suffices to find  $x'_1, x'_2 \in C'$  with  $\text{dir}(x'_1 - x'_2) = v$  for any given  $v \in V \setminus \{0\}$ , so let such a  $v$  be given.

Choose any  $x'_1 \in C'$  and set  $y_1 := \pi(x'_1)$ . Any fiber of  $\pi$  contains exactly one element of  $C$ ; let  $x_1$  be this unique element of  $C \cap \pi^{-1}(y_1)$ . Choose  $y_2 \in \pi(B)$  such that  $\text{dir}(y_1 - y_2) = \bar{\pi}(v)$  and  $v(y_1 - y_2) = v(x'_1 - x_1)$ . Now let  $x_2$  and  $x'_2$  be the images of  $x_1$  and  $x'_1$  under a risometry  $\phi: \pi^{-1}(y_1) \xrightarrow{\sim} \pi^{-1}(y_2)$ .

We have  $x_2 \in C$ , so  $\text{dir}(x_1 - x_2) = v$ . Since  $\phi$  is a risometry, we have  $\text{rv}(x'_1 - x_1) = \text{rv}(x'_2 - x_2)$ , so  $v((x'_1 - x'_2) - (x_1 - x_2)) > v(x'_1 - x_1) = v(x_1 - x_2)$ , which implies  $\text{rv}(x'_1 - x'_2) = \text{rv}(x_1 - x_2)$  and thus  $\text{dir}(x'_1 - x'_2) = v$ .  $\square$

The following lemma is the main tool to prove  $V$ -translatability of a coloring  $\chi$  on a ball  $B \subseteq K^n$  for some  $d$ -dimensional  $V \subseteq k^n$ . Let  $\pi: B \rightarrow K^d$  exhibit  $V$ . The prerequisites are (i) that all  $\pi$ -fibers are risometric, (ii) that the monochromatic pieces of  $\chi$  are sub-affine, (iii) that we have a  $\chi$ -monochromatic piece  $C$  with  $\text{affdir } C = V$ , and (iv) that outside of a  $d$ -dimensional set, we do already have sufficient translatability with respect to a given (partial)  $t$ -stratification. However, in applications of the lemma, we will not be able to ensure (iv) simultaneously with (i) – (iii); therefore, we allow (i) – (iii) to apply to a refinement  $\chi'$  of  $\chi$ , which is enough to get the result.

**Lemma 4.6.** *Suppose that we have definable sets  $C \subseteq B \subseteq K^n$ , colorings  $\chi, \chi': B \rightarrow \text{RV}^{\text{eq}}$ , an integer  $d \in \{1, \dots, n\}$ , a definable partition  $(S_i)_{d \leq i \leq n}$  of  $B$  and a coordinate projection  $\pi: B \rightarrow K^d$ , with the following properties:*

- $B$  is a ball
- $\dim S_i \leq i$ , and for any ball  $B' \subseteq B \setminus S_d$ ,  $((S_i)_i, \chi)$  is sufficiently translatable on  $B'$  (i.e.,  $j$ -translatable if  $B' \subseteq S_{\geq j}$ )
- $\chi'$  is a refinement of  $((S_i)_i, \chi)$  and all monochromatic pieces of  $\chi'$  are sub-affine
- for each pair of points  $x, x' \in \pi(B)$ , there exists a definable risometry  $\pi^{-1}(x) \rightarrow \pi^{-1}(x')$  respecting  $\chi'$
- $C$  is a  $\chi'$ -monochromatic piece whose affine direction space  $V := \text{affdir}(C)$  is exhibited by  $\pi$  (in particular,  $\dim C = \dim V = d$ )

Then  $((S_i)_i, \chi)$  is  $V$ -translatable.

*Proof.* By Lemma 4.5, for any  $\chi'$ -monochromatic piece  $C'$  we have  $V \subseteq \text{affdir}(C')$ . In particular, if  $C' \subseteq S_d$ , then  $\text{affdir}(C') = V$  and  $C'$  is  $V$ -translatable on  $B$  by Lemma 4.2 (2).

**Claim 1.** If  $B' \subseteq B$  is a ball with  $B' \cap S_d = \emptyset$ , then  $V \subseteq W := \text{tsp}_{B'}((S_i)_i)$ .

*Proof of Claim 1.* Set  $d' := \dim W$  and let  $\pi': B' \rightarrow K^{d'}$  be an exhibition of  $W$ . The  $\pi'$ -fibers of  $S_{d'}$  are finite but non-empty. Choose a subball  $B'' \subseteq B'$  such that  $B'' \cap (\pi')^{-1}(x) \cap S_{d'}$  is a singleton for each  $x \in \pi'(B'')$ . Then  $\text{affdir}(S_{d'} \cap B'') = \text{tsp}_{B''}((S_i)_i) = W$ . Now choose any  $\chi'$ -monochromatic piece  $C' \subseteq S_{d'}$  with  $\dim(C' \cap B'') = d'$ . Then  $W \subseteq \text{affdir } C'$  and  $\dim(\text{affdir } C') = \dim C' = d'$  together imply  $W = \text{affdir } C'$ , which contains  $V$  by Lemma 4.5.

If  $S_d = \emptyset$ , then we are done using  $B' = B$ , so from now on suppose  $S_d \neq \emptyset$ .

**Claim 2.** Fix  $x \in \pi(B)$ , let  $F = \pi^{-1}(x)$  be the fiber over  $x$ , and set  $T_i := S_{i+d} \cap F$  for  $i \leq n - d$ . Then  $(T_i)_{i \leq n-d}$  is a  $t$ -stratification of  $F$  reflecting  $\chi|_F$ .

*Proof of Claim 2.* Since all fibers over  $\pi(B)$  are definably risometric, we have  $\dim T_j \leq j$ . Consider a ball  $B' \subseteq B$  intersecting  $F$  and set  $B'_x := B' \cap F$ . We have to show that if  $B'_x \subseteq T_{\geq j}$ , then  $((T_i)_i, \chi)$  is  $j$ -translatable on  $B'_x$ . For  $j = 0$ , there is nothing to do, so suppose  $j \geq 1$ . Then  $B'_x \cap S_d = \emptyset$ , and using that any  $\chi'$ -monochromatic piece  $C'$  contained in  $S_d$  is  $V$ -translatable, we get  $B' \cap S_d = \emptyset$ . Now Claim 1 together with Lemma 3.9 implies  $j$ -translatability on  $B'_x$ .

**Claim 3.**  $((S_i)_i, \chi)$  is  $V$ - $\pi$ -point-translatable.

*Proof of Claim 3.* Let  $x, x' \in \pi(B)$  and  $y \in \pi^{-1}(x)$  be given; we need to find  $y' \in \pi^{-1}(x')$  of the same  $((S_i)_i, \chi)$ -color as  $y$  with  $\text{dir}(y - y') \in V$ . Set  $\delta := v(x - x')$ . If  $B(y, \geq \delta)$  does not intersect  $S_d$ , then  $y'$  is obtained using Claim 1. Otherwise, let  $C' \subseteq S_d$  be a  $\chi'$ -monochromatic piece intersecting  $B(y, \geq \delta)$ , let  $z, z'$  be the unique elements of  $C' \cap \pi^{-1}(x)$  and  $C' \cap \pi^{-1}(x')$ , respectively, and let  $y'$  be the image of  $y$  under a risometry  $\pi^{-1}(x) \rightarrow \pi^{-1}(x')$  respecting  $\chi'$ . Now  $v(y - z) \geq \delta$  and  $\text{rv}(y - z) = \text{rv}(y' - z')$  together imply  $v((y - y') - (z - z')) > \delta$ , and thus  $\text{dir}(y - y') = \text{dir}(z - z') \in V$ .

Now, Claims 2 and 3 (together with  $S_d \neq \emptyset$ ) are all we need to apply Corollary 3.25, which yields the desired  $V$ -translatability.  $\square$

To be able to apply the previous lemma, we need to prove that for  $\pi$  and  $\chi'$  as above, all  $\pi$ -fibers are risometric, when colored with  $\chi'$ . The following lemma offers one way to do that, using 1-translatability of  $\chi'$  on certain affine subspaces.

**Lemma 4.7.** *Let the following be given:*

- a coloring  $\chi: B \rightarrow \text{RV}^{\text{eq}}$  of a ball  $B \subseteq K^n$ ;
- a vector space  $V \subseteq k^n$  of dimension  $d \geq 1$  exhibited by  $\pi: B \rightarrow K^d$ ;
- a  $\chi$ -monochromatic piece  $C \subseteq B$  with  $\text{affdir}(C) \subseteq V$ .

*Suppose that for each coordinate projection  $\rho: K^d \rightarrow K^{d-1}$  and each  $y \in \rho(\pi(B))$ ,  $\chi$  is 1-translatable on the fiber  $\pi^{-1}(\rho^{-1}(y))$ .*

*Then for any  $x_1, x_2 \in \pi(B)$ , there exists a definable risometry  $\pi^{-1}(x_1) \rightarrow \pi^{-1}(x_2)$  respecting  $\chi$ .*

**Remark 4.8.** A posteriori, this implies  $\dim C = d$ , so  $\text{affdir}(C) = V$  and  $C$  is sub-affine.

*Proof of Lemma 4.7.* It is enough to find such risometries  $\pi^{-1}(x_1) \rightarrow \pi^{-1}(x_2)$  under the assumption that  $x_1$  and  $x_2$  differ in only one coordinate and moreover  $\pi^{-1}(x_1)$  intersects  $C$ . Indeed, existence of a risometry implies that  $\pi^{-1}(x_2)$  intersects  $C$ , too, so by repeatedly applying this (starting with a fiber intersecting  $C$  and modifying coordinates one by one), we first get that every fiber intersects  $C$ , and then we obtain risometries between any two fibers by composition.

So suppose now that  $x_1$  and  $x_2$  differ only in one coordinate and let  $\rho: K^d \rightarrow K^{d-1}$  be the coordinate projection satisfying  $\rho(x_1) = \rho(x_2) =: y$ ; let  $F := \pi^{-1}(\rho^{-1}(y)) \subseteq B$  be the corresponding fiber. By assumption, there exists a one-dimensional  $W \subseteq \ker(\bar{\rho} \circ \bar{\pi}) \subseteq k^n$  such that  $\chi$  is  $W$ -translatable on  $F$ . In particular, the non-empty set  $C \cap F$  is  $W$ -translatable, so  $W \subseteq V$  by Lemma 4.2 (1). Since  $\dim(\ker(\bar{\rho} \circ \bar{\pi}) \cap V) = 1$ ,  $W$  is equal to this intersection, so  $\pi|_F$  exhibits  $W$ . From a translator  $(\alpha_x)_{x \in \pi(F-F)}$  of  $\chi|_F$  with respect to  $\pi$ , we obtain a risometry  $\phi: \pi^{-1}(x_1) \rightarrow \pi^{-1}(x_2)$  by restricting  $\alpha_{x_2-x_1}$  to  $\pi^{-1}(x_1)$ .  $\square$

**4.3. The big induction.** This subsection contains the actual proof of the main theorem. We first have to prove the case  $n = 1$  separately.

**Lemma 4.9.** *For every  $\emptyset$ -definable coloring  $\chi: K \rightarrow \text{RV}^{\text{eq}}$ , there exists a finite  $\emptyset$ -definable set  $T_0 \subseteq K$  such that each ball  $B \subseteq K$  not intersecting  $T_0$  is monochromatic.*

*Proof.* Apply b-minimality to a  $\chi$ -monochromatic piece  $C := \chi^{-1}(\sigma) \subseteq K$ . We get a  $\sigma$ -definable auxiliary set  $S_\sigma$  and  $\sigma$ -definable functions  $c_\sigma: S_\sigma \rightarrow K$  and  $\xi_\sigma: S_\sigma \rightarrow \text{RV}$  such that the sets  $C(\sigma, s) := c_\sigma(s) + \text{rv}^{-1}(\xi_\sigma(s))$  form a partition of  $C$ . By compactness, this can be done uniformly in  $\sigma$ . By Hypothesis 2.8 (2), the map  $(\sigma, s) \mapsto c_\sigma(s)$  has finite image; let  $T_0$  be this image. Then for any ball  $B \subseteq K \setminus T_0$  and any  $\sigma, s$ ,  $B$  is either entirely contained in  $C(\sigma, s)$  or it is disjoint from  $C(\sigma, s)$ . This implies the lemma.  $\square$

Now we are ready for the main theorem. Concerning its statement, recall that by a ball  $B_0 \subseteq K^n$ , we mean either an open or a closed ball (in the “maximum metric”) and that we regard  $K^n$  as an open ball. Also recall that a coloring is simply a definable map into any auxiliary sort, i.e., any sort of  $\text{RV}^{\text{eq}}$ . Finally, recall that colorings of  $B_0$  and auxiliary-parametrized definable families of subsets of  $B_0$  are essentially the same by Lemma 2.22.

**Theorem 4.10.** *Let  $\mathcal{L}$  be an expansion of the valued field language  $\mathcal{L}_{\text{Hen}}$  and let  $K$  be an  $\mathcal{L}$ -structure whose theory satisfies Hypothesis 2.8. (In particular,  $K$  is a Henselian valued field of equi-characteristic 0.) Then, for every  $\emptyset$ -definable ball  $B_0 \subseteq K^n$  and every  $\emptyset$ -definable coloring  $\chi: B_0 \rightarrow \text{RV}^{\text{eq}}$ , there exists an  $\emptyset$ -definable  $t$ -stratification  $(S_i)_{i \leq n}$  of  $B_0$  reflecting  $\chi$ .*

*Proof.* We do a big induction on  $n$ , i.e., we assume that the theorem holds for all smaller  $n$ .

By extending  $\chi$  trivially outside of  $B_0$ , we may suppose  $B_0 = K^n$ .

By decreasing induction on  $d$ , we prove the following.

- ( $\star_d$ ) There exists a  $\emptyset$ -definable partition  $(S_i)_{d \leq i \leq n}$  of  $K^n$  with  $\dim S_i \leq i$  such that for any ball  $B \subseteq S_{>d}$ ,  $((S_i)_i, \chi)$  is sufficiently translatable on  $B$ .

Note that ( $\star_0$ ) implies the theorem. (For balls intersecting  $S_0$ , there is nothing to prove.)

The start of induction ( $\star_n$ ) is trivial (set  $S_n = K^n$ ). Now suppose that  $(S_i)_{d \leq i \leq n}$  is given such that ( $\star_d$ ) holds (for some  $d \geq 1$ ). It suffices to find a set  $S_{d-1}$  of dimension at most  $d-1$  such that on any ball  $B \subseteq K^n \setminus S_{d-1}$ ,  $((S_i)_i, \chi)$  is sufficiently translatable; after that, we obtain ( $\star_{d-1}$ ) using the partition  $S_{d-1}, (S_i \setminus S_{d-1})_{i \geq d}$ . Moreover, by induction it is enough to check translatability on balls  $B$  with  $B \cap S_d \neq \emptyset$ .

We have to do the case  $d = 1$  separately.

**The case  $d \geq 2$ :**

First, we choose a refinement  $\chi'$  of  $((S_i)_i, \chi)$  whose monochromatic pieces are sub-affine (using Lemma 4.4). Now consider a coordinate projection  $\pi: K^n \twoheadrightarrow K^{d-1}$ . By induction on  $n$  (and using  $d \geq 2$ ), we can find  $t$ -stratifications of the fibers of  $\pi$  reflecting  $\chi'$  on the fibers. Taking the union of corresponding strata of different fibers yields a  $\emptyset$ -definable partition  $(T_i)_{i \leq n-d+1}$  of  $K^n$  with  $\dim T_i \leq i+d-1$ . Define  $S_{d-1}$  to be the union of the sets  $T_0$  for all coordinate projections  $\pi: K^n \twoheadrightarrow K^{d-1}$ .

Now let a ball  $B \subseteq K^n \setminus S_{d-1}$  with  $B \cap S_d \neq \emptyset$  be given; we have to prove that  $((S_i)_i, \chi)$  is  $d$ -translatable on  $B$ . We will do this by applying Lemma 4.6 to  $B$ ,  $\chi$ ,  $\chi'$ , and  $(S_i)_i$ ; now let us produce the remaining ingredients.

Let  $C \subseteq S_d$  be any  $\chi'$ -monochromatic piece intersecting  $B$ , let  $V \subseteq k^n$  be  $d$ -dimensional such that  $\text{affdir}(C) \subseteq V$  (which exists since  $\dim C \leq d$ ), and choose an exhibition  $\pi: B \rightarrow K^d$  of  $V$ . The only missing prerequisite for Lemma 4.6 is now that there exists a definable risometry preserving  $\chi'$  between any two  $\pi$ -fibers  $\pi^{-1}(x)$ ,  $\pi^{-1}(x')$ ; this then also implies  $\dim C = d$  and hence  $\text{affdir}(C) = V$ .

To get the risometries between the fibers, we apply Lemma 4.7 to  $\chi'$ ,  $V$ , and  $C$ . Suppose that  $\rho: K^d \rightarrow K^{d-1}$  is a coordinate projection and  $F$  is a fiber of  $\pi' := \rho \circ \pi$ . Consider the partition  $(T_i)_{i \leq n-d+1}$  of  $K^n$  obtained from t-stratifications of the fibers of  $\pi'$  in the above definition of  $S_{d-1}$ . Since  $T_0 \subseteq S_{d-1}$  we have  $B \cap T_0 = \emptyset$ , so in particular,  $\chi'|_F$  is 1-translatable on  $B \cap F$ , which is what we need for Lemma 4.7.

**The case  $d = 1$ :**

Recall that we do already have a partition  $(S_i)_{i \geq 1}$  which is good outside of  $S_1$ . We will now carry out an additional induction, during which the bad set will become “more and more 0-dimensional”. More precisely, consider the following statement for  $e \in \{0, \dots, n\}$ .

- ( $\star\star_e$ ) There exists a family of definable sets  $X_\rho$  parametrized by the coordinate projections  $\rho: K^n \rightarrow K^e$ , such that  $\rho(X_\rho)$  is finite,  $\dim X_\rho \leq 1$ , and on any ball  $B \subseteq K^n \setminus \bigcup_\rho X_\rho$ ,  $((S_i)_i, \chi)$  is sufficiently translatable.

Write  $X := \bigcup_\rho X_\rho$  for the union. The statement ( $\star\star_0$ ) follows from ( $\star_1$ ), since we can take  $X = X_\rho = S_1$  (where  $\rho: K^n \rightarrow K^0$ ). The statement ( $\star\star_n$ ) is what we want to prove, since in that case,  $\rho = \text{id}_{K^n}$  implies that  $X$  itself is finite, so we can set  $S_0 = X$ . Thus it remains to prove  $(\star\star_e) \Rightarrow (\star\star_{e+1})$  for  $0 \leq e < n$ . Let  $X = \bigcup_\rho X_\rho$  be given for  $e$ , and let us construct a set  $X'$  for  $e+1$ . We start by choosing a refinement  $\chi'$  of  $((S_i)_i, \chi, (X_\rho)_\rho)$  whose monochromatic pieces are sub-affine.

Let  $\rho: K^n \rightarrow K^e$  and  $\pi: K^n \rightarrow K$  be coordinate projections “projecting to different coordinates”, i.e., such that  $(\rho, \pi): K^n \rightarrow K^e \times K$  is surjective.

By the main induction on  $n$ , we can find t-stratifications of the fibers of  $\pi$  reflecting  $\chi'$  on the fibers. By Proposition 3.23 (2) and (3), there exists a coloring  $\chi_0: K \rightarrow \text{RV}^{\text{eq}}$  such that for any  $\chi_0$ -monochromatic set  $C_0 \subseteq K$ , we have a definable compatible family of risometries between the fibers  $\pi^{-1}(x)$  for  $x$  running through  $C_0$ . Lemma 4.9 yields a finite subset  $T_0 \subseteq K$  such that any ball  $B' \subseteq K \setminus T_0$  is  $\chi_0$ -monochromatic. Recall that  $\pi^\vee: K^n \rightarrow K^{n-1}$  denotes the “complement” of  $\pi$  and define the set  $X_{\rho, \pi}$  as follows:

$$X_{\rho, \pi} := \{x \in \pi^{-1}(T_0) \mid \pi^\vee(x) \in \pi^\vee(X_\rho)\}.$$

We define  $X'$  to be the union of all such  $X_{\rho, \pi}$  (for all  $\rho, \pi$  as above).

Since  $T_0$  is finite,  $\dim X_{\rho, \pi} \leq \dim X_\rho \leq 1$  and  $(\rho, \pi)(X_{\rho, \pi})$  is finite, so it remains to check that on a ball  $B \subseteq K^n \setminus X'$ ,  $((S_i)_i, \chi)$  is sufficiently translatable. If  $B \cap X = \emptyset$ , then we know this by induction, so suppose that  $B \cap X_\rho \neq \emptyset$  for some  $\rho: K^n \rightarrow K^e$ .

Let  $C \subseteq X_\rho$  be a monochromatic piece of  $\chi'$  with  $C \cap B \neq \emptyset$ . If  $\dim(C \cap B) = 0$ , then let  $\pi: K^n \rightarrow K$  be any coordinate projection projecting to a different

coordinate than  $\rho$ . Otherwise, set  $V := \text{affdir}(C)$  and let  $\pi$  be an exhibition of  $V$ . Since  $\rho(C)$  is finite, we have  $V \subseteq \ker \bar{\rho}$ , so in this case too,  $\rho$  and  $\pi$  project to different coordinates.

Let  $\chi_0, T_0$  be as in the construction of  $X_{\rho, \pi}$ . Then  $\pi(B) \cap T_0 = \emptyset$ , since otherwise, for  $x \in \pi(B) \cap T_0$  and  $y \in B \cap X_\rho$ , the point  $y' \in K^n$  with  $\pi(y') = x$  and  $\pi^\vee(y') = \pi^\vee(y)$  lies both in  $B$  and in  $X_{\rho, \pi}$ , contradicting  $B \cap X_{\rho, \pi} = \emptyset$ . By our choice of  $T_0$ , this implies that  $\pi(B)$  is  $\chi_0$ -monochromatic and thus all fibers  $\pi^{-1}(x)$  (for  $x \in \pi(B)$ ) are risometric when colored with  $\chi'$ . In particular,  $C$  intersects every fiber and thus  $\dim(C \cap B) = 1$ .

Now we can apply Lemma 4.6 to  $C \cap B$ ,  $B$ ,  $\chi$ ,  $\chi'$ ,  $\pi$ , and the partition  $(S_1 \cup X, (S_i \setminus X)_{i \geq 2})$  (restricted to  $B$ ); this yields that  $((S_i)_i, \chi)$  is  $V$ -translatable on  $B$ , which is what we had to show.  $\square$

**4.4. Corollaries.** Using compactness, we can deduce a version of the main theorem which works uniformly for all models of our theory  $\mathcal{T}$  and in fact also for all models of a finite subset of  $\mathcal{T}$ , provided that the notion of t-stratification makes sense. In particular, we also get t-stratifications in all Henselian valued fields of sufficiently big residue characteristic (both, in the equi-characteristic and the mixed characteristic case). Note that in equi-characteristic, there is no good notion of dimension of a definable set; there, “ $\dim S_i = i$ ” means that we stupidly apply Definition 2.13. However, in the case of the pure valued field language, this problem will be solved in Corollary 5.9: we will see that we can choose the t-stratification such that each set  $S_{\leq i}$  is Zariski closed and has dimension  $i$  in the algebraic sense.

**Corollary 4.11.** *Suppose  $\mathcal{T}$  is an  $\mathcal{L}$ -theory satisfying Hypothesis 2.8. Let  $\chi$  be an  $\mathcal{L}$ -formula defining a coloring of  $K^n$  (for any model  $K \models \mathcal{T}$ ). Then there exist  $\mathcal{L}$ -formulas  $\psi_0, \dots, \psi_n$  and a finite subset  $\mathcal{T}_0 \subseteq \mathcal{T}$  such that for each model  $K$  of  $\mathcal{T}_0$ ,  $(\psi_i(K))_i$  is a t-stratification reflecting the coloring  $\chi(K)$ . (For this to make sense, we assume that  $\mathcal{T}_0$  in particular says that  $K$  is a valued field.)*

*Proof.* By Theorem 4.10, we find formulas  $(\psi_i)_i$  defining a t-stratification for any fixed model  $K \models \mathcal{T}$ . Moreover, by Corollary 3.26, we also find a formula  $\eta$  (depending on  $(\psi_i)_i$ ) defining the corresponding straighteners on all balls  $B \subseteq K^n$  (using parameters). This allows us to formulate a first order sentence which holds in an  $\mathcal{L}$ -structure  $K'$  iff  $(\psi_i(K'))_i$  is a t-stratification reflecting  $\chi(K')$ :

- ( $\star$ ) For each  $i$ ,  $\psi_i(K')$  is either empty or has dimension  $i$  in the sense of Definition 2.13, and  
for each ball  $B \subseteq (K')^n$ , there exists a parameter  $b$  such that  $\eta(K', b)$  defines a straightener on  $B$  which proves that  $((\psi_i(K'))_i, \chi(K'))$  is sufficiently translatable on  $B$ .

By compactness,  $\psi_i$  and  $\eta$  can be chosen such that ( $\star$ ) holds in all models of  $\mathcal{T}$ . Moreover, then ( $\star$ ) follows already from a finite subset of  $\mathcal{T}$ .  $\square$

**Corollary 4.12.** *Let  $\chi_q: K^n \rightarrow \text{RV}^{\text{eq}}$  be a definable family of colorings, parametrized by  $q \in Q$  (for some definable set  $Q$  in any sort). Then there exists a coloring  $\chi': Q \rightarrow \text{RV}^{\text{eq}}$  such that  $\chi'(q_1) = \chi'(q_2)$  implies that there exists a  $(q_1, q_2)$ -definable risometry  $\phi: K^n \rightarrow K^n$  with  $\chi_{q_1} \circ \phi = \chi_{q_2}$ . This also works uniformly for all models  $K$  of a finite subset of  $\mathcal{T}$ .*

*Proof.* Add a constant symbol for  $q$  to the language. Then Corollary 4.11 yields uniformly defined  $t$ -stratifications reflecting  $\chi_q$  in each model of a finite subset of  $\mathcal{T}$  and for each  $q \in Q$ . Now  $\chi'$  is obtained from Proposition 3.23 (2).  $\square$

In the next section (where we prove an algebraic version of the main result), we will give an algebraic version of this corollary (Corollary 5.11). (That version follows directly from Corollary 4.12, but thematically, it fits better into the next section.)

## 5. ALGEBRAIC RESULT

Up to now, for a  $t$ -stratification  $(S_i)_i$  we know that the sets  $S_{\leq i}$  are closed in the valued field topology. However, in a purely algebraic setting, it would be natural to require the sets  $S_{\leq i}$  to be Zariski closed. We will now show that indeed this can be achieved (Corollary 5.9). More generally, for any language (satisfying Hypothesis 2.8), Proposition 5.4 says that we can obtain sets  $S_{\leq i}$  which are closed in any given topology satisfying some suitable conditions.

**5.1. Getting closed sets  $S_{\leq i}$ .** The goal of this subsection is to prove Proposition 5.4 and its uniform version Proposition 5.6. We start with some preliminary lemmas.

**Lemma 5.1.** *If  $(S_i)_i$  is a  $t$ -stratification of  $K^n$  reflecting a definable set  $X \subseteq K^n$ , then  $X \subseteq S_{\leq \dim X}$ .*

*Proof.* Follows from Lemma 3.10.  $\square$

**Definition 5.2.** A  $t$ -stratification  $(S'_i)_i$  is a *refinement* of a  $t$ -stratification  $(S_i)_i$  if any of the following equivalent conditions holds:

- (1) The rainbow of  $(S'_i)_i$  refines the rainbow of  $(S_i)_i$ .
- (2) Any coloring reflected by  $(S_i)_i$  is also reflected by  $(S'_i)_i$
- (3)  $(S'_i)_i$  reflects the coloring  $(S_i)_i$

*Proof of equivalence.* (1)  $\iff$  (2) follows from Proposition 3.20 (1)  $\iff$  (2); (3)  $\Rightarrow$  (2) can be obtained using Proposition 3.20 (1)  $\iff$  (3); and (1) implies that the rainbow of  $(S'_i)_i$  refines the coloring  $(S_i)_i$ , so (3) follows from Proposition 3.20 (2)  $\Rightarrow$  (1).  $\square$

There is a slight clash of nomenclature here: if we view both  $(S_i)_i$  and  $(S'_i)_i$  as colorings, then refinement means something different; however, it should always be clear from the context in which sense “refinement” is meant.

If  $(S'_i)_i$  is a refinement of  $(S_i)_i$ , then  $S_{\leq i} \subseteq S'_{\leq i}$  for all  $i$  (by Lemma 5.1); however, the converse is not true in general.

**Lemma 5.3.** *Suppose that we have a definable set  $X \subseteq K^n$  with  $\dim X \leq d$  and two  $t$ -stratifications  $(S_i)_i, (T_i)_i$  of  $K^n$ , where  $(T_i)_i$  refines  $(S_i)_i$  and reflects  $X$ . Then the following defines a  $t$ -stratification which also refines  $(S_i)_i$  and reflects  $X$ :  $S'_{\leq i} := T_{\leq i}$  for  $i < d$  and  $S'_{\leq i} := S_{\leq i} \cup X \cup T_{\leq d-1}$  for  $i \geq d$ .*

In other words, this lemma allows us to refine  $(S_i)_i$  to  $(S'_i)_i$  such that  $(S'_i)_i$  reflects  $X$  and such that  $(S_i)_i$  and  $(S'_i)_i$  agree outside of  $X \cup T_{\leq d-1}$ .

*Proof.* It is clear that  $\dim S'_{\leq i} \leq i$ , so now consider a ball  $B \subseteq S'_{\geq j}$ ; we have to show that  $((S'_i)_i, (S_i)_i, X)$  is  $j$ -translatable on  $B$ . If  $j \leq d$ , then we have  $j$ -translatability since  $B \subseteq S'_{\geq j} = T_{\geq j}$ ; if  $j \geq d+1$ , then  $B \cap (X \cup T_{\leq d-1}) = \emptyset$  and  $S'_i \cap B = S_i \cap B$  for every  $i$ , so  $j$ -translatability follows from  $j$ -translatability of  $(S_i)_i$ .  $\square$

**Proposition 5.4.** *Suppose that we have topology  $\tau$  on  $K^n$  such that for any definable set  $X \subseteq K^n$ , the closure  $X^\tau$  is also definable and  $\dim X^\tau = \dim X$ . Then any  $t$ -stratification  $(S_i)_i$  can be refined to a  $t$ -stratification  $(S'_i)_i$  such that  $S'_{\leq i}$  is  $\tau$ -closed for each  $i$ .*

Note that we do not require  $\dim(X^\tau \setminus X) < \dim X$ .

*Proof.* Set  $\partial X := X^\tau \setminus X$ . Suppose that for some given  $d \in \{0, \dots, n\}$ ,  $(S_i)_i$  satisfies  $\dim \partial S_{\leq i} \leq d$  for all  $i$ . From this, we construct a refinement  $(S'_i)_i$  satisfying  $\dim \partial S'_{\leq i} \leq d-1$ . Applying this repeatedly yields the proposition (where  $\dim X \leq -1$  will mean  $X = \emptyset$ ). So let  $d$  be given as above.

For  $i$  from  $n$  to  $0$ , define recursively  $D_i := \partial(S_{\leq i} \cup D_{i+1} \cup \dots \cup D_n)$ . Inductively, we get  $\dim D_i \leq d$ . Set  $D := \bigcup_{i=0}^n D_i$ , choose any  $t$ -stratification  $(T_i)_i$  reflecting  $((S_i)_i, D)$ , and apply Lemma 5.3 to  $D$ ,  $d$ ,  $(S_i)_i$  and  $(T_i)_i$ ; we claim that the resulting  $t$ -stratification  $(S'_i)_i$  satisfies  $\partial S'_{\leq i} \subseteq (S'_{\leq d-1})^\tau$  (which implies  $\dim \partial S'_{\leq i} \leq d-1$  and  $\partial S'_{\leq i} = \emptyset$  if  $d = 0$ ).

For  $i \leq d-1$ , there is nothing to prove, so suppose  $i \geq d$ . The set  $S'_{\leq i} = S_{\leq i} \cup D \cup T_{\leq d-1}$  can be written as  $\bigcup_{j \leq i} (S_{\leq j} \cup D_j \cup \dots \cup D_n) \cup T_{\leq d-1}$ . Each set  $S_{\leq j} \cup D_j \cup \dots \cup D_n$  is closed by definition of  $D_j$ , so we get  $\partial S'_{\leq i} \subseteq \partial T_{\leq d-1} = \partial S'_{\leq d-1}$ .  $\square$

To find  $t$ -stratifications where each set  $S_{\leq i}$  is given as a zero set of some polynomials, we will need a variant of Proposition 5.4 which works uniformly for all models of  $\mathcal{T}$ . (With the present version of Proposition 5.4 we can get Zariski closed sets in each model  $K \models \mathcal{T}$  individually, but if we would use compactness to get uniform formulas for all models of  $\mathcal{T}$ , then these formulas would not be polynomials anymore.) Thus here is such a uniform version.

**Definition 5.5.** For an  $\mathcal{L}$ -formula  $\phi$  whose free variables live in the valued field sort, set  $\dim \phi := \max_{K \models \mathcal{T}} \dim \phi(K)$ .

**Proposition 5.6.** *In the following, all  $\mathcal{L}$ -formulas have  $n$  free valued field variables, and “ $\phi \rightarrow \psi$ ” means “ $\mathcal{T} \vdash \forall x (\phi(x) \rightarrow \psi(x))$ ”.*

*Suppose that we have a family  $\Delta$  of  $\mathcal{L}$ -formulas with the following properties:*

- (1)  $\Delta$  is closed under conjunctions and disjunctions
- (2) For each  $\mathcal{L}$ -formula  $\phi$ , there exists a minimal formula  $\phi^\Delta \in \Delta$  with  $\phi \rightarrow \phi^\Delta$ ; minimal means: for any other  $\psi \in \Delta$  with  $\phi \rightarrow \psi$ , we have  $\phi^\Delta \rightarrow \psi$ .
- (3) For each  $\mathcal{L}$ -formula  $\phi$ , we have  $\dim \phi^\Delta = \dim \phi$ .

*Then for any tuple of formulas  $(\phi_i)_i$  defining a  $t$ -stratification in every model of  $\mathcal{T}$ , we can find a tuple of formulas  $(\phi'_i)_i$  refining these  $t$ -stratifications such that for each  $i$ ,  $\phi'_0 \vee \dots \vee \phi'_i$  is equivalent to a formula of  $\Delta$ .*

*Proof.* Rewrite the proof of Proposition 5.4 using formulas instead of definable sets in a fixed model. In particular, instead of choosing a  $t$ -stratification  $(T_i)_i$  in a single model  $K$ , use Corollary 4.11 to find formulas  $\psi_i$  defining a  $t$ -stratification in every model.  $\square$

**5.2. Algebraic strata.** Let us now work in the pure valued field language  $\mathcal{L}_{\text{Hen}}$ . To get t-stratifications with Zariski closed sets  $S_{\leq i}$ , it remains to check that the family of formulas which are conjunctions of polynomial equations satisfies the prerequisites of Proposition 5.6. This then yields an almost purely algebraic formulation of the main Theorem—only “almost”, since in the definition of t-stratification, we require the straighteners to be definable. (Of course, this condition can simply be omitted, but this weakens the result.)

Fix an integral domain  $A$  of characteristic 0. We set  $\mathcal{L} := \mathcal{L}_{\text{Hen}}(A)$  and  $\mathcal{T} := \mathcal{T}_{\text{Hen}} \cup \{\text{positive atomic diagram of } A \text{ in } \mathcal{L}_{\text{ring}}\}$ ; in other words, models  $K$  of  $\mathcal{T}$  are Henselian valued fields of equi-characteristic 0 together with a ring homomorphism  $A \rightarrow K$ . (This is done mainly to fit to the usual language of algebraic geometry.) We fix  $n \in \mathbb{N}$  and let  $\Delta$  be the set of conjunctions of polynomial equations with coefficients in  $A$ . For any model  $K \models \mathcal{T}$ , this yields a topology on  $K^n$  whose closed sets are  $\phi(K)$ ,  $\phi \in \Delta$ . We call this the Zariski topology and we denote the Zariski closure of a set  $X \subseteq K^n$  by  $X^{\text{Zar}}$ . (Note that this topology depends on  $A$ .) In terms of algebraic geometry, formulas in  $\Delta$  correspond to Zariski closed subsets of the scheme  $\mathbb{A}_A^n$ , and our topology on  $K^n$  is the one which the Zariski topology on  $\mathbb{A}_A^n$  induces on the  $K$ -valued points  $\mathbb{A}_A^n(K)$ .

Note that for  $\phi \in \Delta$ , we have two notions of dimension: the one given in Definition 5.5 and the algebraic one, where we consider  $\phi$  as a variety over  $A$ . However, by considering an algebraically closed model of  $\mathcal{T}$ , we see that the two notions of dimension coincide.

Given an arbitrary  $\mathcal{L}$ -formula  $\phi$ , it is clear that there exists well-defined “Zariski closure” of  $\phi$ , i.e., a minimal formula  $\phi^\Delta \in \Delta$  implied by  $\phi$ . To be able to apply Proposition 5.6, it remains to check that  $\phi^\Delta$  has the same dimension as  $\phi$ . This has been proven in [10] or [5] for example, but in slightly different contexts than ours, so let us quickly repeat the proof from [10]. We first work in a fixed model  $K$ .

**Lemma 5.7.** *For every  $\emptyset$ -definable set  $X \subseteq K^n$ , there exists a formula  $\psi \in \Delta$  such that  $X \subseteq \psi(K)$  and  $\dim \psi = \dim X$ .*

*Proof.* In this proof, we will write  $\text{rv}^\ell$  for the canonical map  $K^\ell \rightarrow \text{RV}^\ell$  (in contrast to the map  $\text{rv}: K^\ell \rightarrow \text{RV}^{(\ell)}$  mainly used in the remainder of the article).

By quantifier elimination (see e.g. [3], Theorem 6.3.7),  $X$  is of the form

$$X = \{x \in K^n \mid (\text{rv}(f_1(x)), \dots, \text{rv}(f_\ell(x))) \in \Xi\} = f^{-1}((\text{rv}^\ell)^{-1}(\Xi))$$

where  $f = (f_1, \dots, f_\ell)$  is an  $\ell$ -tuple of polynomials with coefficients in  $A$  and  $\Xi \subseteq \text{RV}^\ell$  is  $\emptyset$ -definable. The statement of the lemma is closed under finite unions, so we can do a case distinction on whether  $f_i(x) = 0$  or not for each  $i$ ; in other words,  $X$  is of the form

$$X = \psi(K) \cap f^{-1}((\text{rv}^\ell)^{-1}(\Xi))$$

for  $\psi \in \Delta$ ,  $f$  as above and  $\Xi \subseteq (\text{RV} \setminus \{0\})^\ell$ .

Write  $\tilde{K}$  for the algebraic closure of  $K$ . We may assume that  $\psi(\tilde{K})$  is the Zariski closure of  $X$  in  $\tilde{K}^n$ . In particular,  $X$  contains a regular point  $x$  of  $\psi(\tilde{K})$ , i.e., on a Zariski-neighborhood of  $x$ ,  $\psi(\tilde{K})$  is defined by  $n - \dim \psi$  polynomials and the Jacobian matrix at  $x$  of this tuple of polynomials has maximal rank.

Now  $(\text{rv}^\ell)^{-1}(\Xi)$  is open in the valuation topology, so in that topology, there is a neighborhood  $U \subseteq \psi(K)$  of  $x$  which is contained in  $X$ . Using the implicit function

theorem and regularity at  $x$ , we find a coordinate projection  $\pi: K^n \twoheadrightarrow K^{\dim \psi}$  such that  $\pi(U)$  contains a ball in  $K^{\dim \psi}$ . This implies  $\dim X = \dim \psi$ .  $\square$

Now we make the result uniform for all models of  $\mathcal{T}$ :

**Lemma 5.8.** *For every  $\mathcal{L}$ -formula  $\phi$  in  $n$  valued field variables, there exists a formula  $\psi \in \Delta$  with  $\dim \psi = \dim \phi$  and  $\phi(K) \subseteq \psi(K)$  for all models  $K \models \mathcal{T}$ .*

*Proof.* For each  $K$  separately, Lemma 5.7 yields a formula  $\psi_K \in \Delta$  with  $\phi(K) \subseteq \psi_K(K)$  and  $\dim \psi_K = \dim \phi(K)$ . By compactness, there exists a finite disjunction  $\psi$  of some of the  $\psi_K$  such that  $\phi(K) \subseteq \psi(K)$  for all  $K$ . Since

$$\dim \psi \leq \max_K \dim \psi_K = \max_K \dim \phi(K) = \dim \phi,$$

we are done.  $\square$

Now Proposition 5.6 can be applied to the Zariski topology and we get  $t$ -stratifications such that each set  $S_{\leq i}$  is defined by a conjunction of polynomials (uniformly for all models). Moreover, using that being a  $t$ -stratification is first order in the sense of Corollary 4.11, the same  $t$ -stratification also works in models of a finite subset of  $\mathcal{T}$ . Here is the precise result.

**Corollary 5.9.** *Let  $A$  be an integral domain of characteristic 0,  $\mathcal{L} = \mathcal{L}_{\text{Hen}}(A)$ , and  $\mathcal{T}$  the theory of Henselian valued fields  $K$  of equi-characteristic 0 together with a ring homomorphism  $A \rightarrow K$ . For every  $\mathcal{L}$ -formula  $\psi$  defining a coloring  $K^n \rightarrow \text{RV}^{\text{eq}}$  for any valued field  $K$ , there exists a finite subset  $\mathcal{T}_0 \subseteq \mathcal{T}$  and formulas  $(\phi_i)_i$  such that:*

- *Either  $\dim \phi_i = i$  (in the sense of Definition 5.5) or  $\phi_i$  defines the empty set.*
- *$\phi_0 \vee \dots \vee \phi_d$  is equivalent to a conjunction of polynomial equations with coefficients in  $A$ .*
- *For every model  $K \models \mathcal{T}_0$ ,  $(\phi_i(K))_i$  is a  $t$ -stratification reflecting  $\psi(K)$ .*

(As in Corollary 4.11, we assume that models of  $\mathcal{T}_0$  are valued fields for the statements to make sense.)

Here is an algebraic formulation of the previous corollary; by a “sub-variety of  $\mathbb{A}_A^n$ ”, we simply mean a reduced (not necessarily irreducible) subscheme. Since the notion of a general coloring is not so algebraic, we formulate the theorem for colorings given by a finite family  $(X_\nu)_\nu$  of sub-varieties of  $\mathbb{A}_A^n$ .

**Theorem 5.10.** *Let  $A$  be an integral domain of characteristic 0 and let  $X_\nu$  be sub-varieties of  $\mathbb{A}_A^n$  for  $\nu = 1, \dots, \ell$ . Then there exists an integer  $N \in \mathbb{N}$  and a partition of  $\mathbb{A}_A^n$  into sub-varieties  $S_i$  with the following properties:*

- (1)  $\dim S_i = i$  or  $S_i = \emptyset$
- (2) Each  $S_{\leq i}$  is Zariski closed in  $\mathbb{A}_A^n$ .
- (3) *For every Henselian valued field  $K$  over  $A$  of residue characteristic either 0 or at least  $N$ ,  $(S_i(K))_i$  is a  $t$ -stratification of  $K^n$  reflecting the family of sets  $(X_\nu(K))_\nu$ , i.e., for any ball  $B \subseteq S_{\geq d}(K)$ ,  $(S_d(K), \dots, S_n(K), X_1(K), \dots, X_\ell(K))$  is  $d$ -translatable on  $B$ .*

For the reader who just jumped to this theorem without reading anything else, let me recall that  $d$ -translatability has been defined for colorings in Definition 3.1 and that Subsection 2.8 explains how a tuple  $(Y_j)_j$  of subsets of  $K^n$  is treated as a coloring.

In the next section, we will prove that a  $t$ -stratification  $(S_i)_i$  as in Theorem 5.10 automatically induces a Whitney stratification  $(S_i(\mathbb{C}))_i$  of  $\mathbb{C}^n$  fitting to each  $X_\nu(\mathbb{C})$  in the classical sense (for any ring homomorphism  $A \rightarrow \mathbb{C}$ ), and similarly for  $\mathbb{C}$  replaced by  $\mathbb{R}$  (see Theorem 6.11). In particular, each  $S_i$  is smooth over the fraction field of  $A$ .

To finish this section, let us consider an algebraic formulation of Corollary 4.12 about how the risometry type can vary in a uniform family.

**Corollary 5.11.** *Let  $A$  be an integral domain of characteristic 0, let  $Q$  be any affine variety over  $A$ , and let  $X_\nu$  be sub-varieties of  $\mathbb{A}_Q^n$  for  $\nu = 1, \dots, \ell$ . Then there exists an integer  $N \in \mathbb{N}$  and algebraic maps  $f_1, \dots, f_m: Q \rightarrow \mathbb{A}_A^1$  such that for every Henselian valued field  $K$  over  $A$  of residue characteristic either 0 or at least  $N$ , we have the following.*

*Given  $q \in Q(K)$ , write  $X_{\nu,q} = X_\nu \times_Q \text{spec } K$  for the fiber of  $X_\nu$  over  $q$  and consider  $X_{\nu,q}(K)$  as a subset of  $K^n$ . If two elements  $q, q' \in Q(K)$  satisfy  $\text{rv}(f_\mu(q)) = \text{rv}(f_\mu(q'))$  for all  $\mu$ , then there exists a risometry  $K^n \rightarrow K^n$  sending  $X_{\nu,q}(K)$  to  $X_{\nu,q'}(K)$  for each  $\nu$ .*

*Proof.* Let us fix an embedding  $Q \hookrightarrow \mathbb{A}_A^\ell$ . Applying Corollary 4.12 yields an integer  $N$  and a formula  $\phi$ , such that for every  $K$  as above,  $\phi$  defines a coloring  $\phi_K$  of  $Q(K)$  such that  $\phi(q) = \phi(q')$  implies existence of a risometry as above for  $q, q' \in Q(K)$ . By quantifier elimination [3, Theorem 6.3.7], we may refine  $\phi$  such that it obtains the form  $q \mapsto (\text{rv}(f_1(q)), \dots, \text{rv}(f_m(q)))$  for some polynomials  $f_i$ , which implies the claim.  $\square$

## 6. OBTAINING CLASSICAL WHITNEY STRATIFICATIONS

The main result of this section is that the existence of  $t$ -stratifications implies the existence of classical Whitney stratifications. More precisely, a non-standard model of  $\mathbb{R}$  or  $\mathbb{C}$  can be considered as a valued field, and a partition of the standard model which induces a  $t$ -stratification in the non-standard model is already a Whitney stratification. We will start by proving that  $t$ -stratifications satisfy a kind of analogue of Whitney's Condition (b) (Corollary 6.6). This needs the following additional (very natural) Hypothesis on the language  $\mathcal{L}$  (and the theory  $\mathcal{T}$ ).

**Hypothesis 6.1.** In this section, we require that the residue field is orthogonal to the value group, i.e., any definable set  $X \subseteq k^n \times \Gamma^m$  (in any model of  $\mathcal{T}$ ) is a finite union of sets of the form  $Y_i \times Z_i$ , for some definable sets  $Y_i \subseteq k^n$  and  $Z_i \subseteq \Gamma^m$ .

**Proposition 6.2.** *Any Henselian valued field  $K$  with analytic structure in the sense of [3] satisfies Hypothesis 6.1.*

*Proof.* By quantifier elimination [3, Theorem 6.3.7], any definable subset of  $\text{RV}^n$  can be defined in the restriction to  $\text{RV}$  of the language  $\mathcal{L}_{\text{Hen}, \mathcal{A}}$  from [3]. To that language, add the sorts  $k$  and  $\Gamma$  and a splitting  $\text{RV} \setminus \{0\} \rightarrow k^\times$  of the sequence  $k^\times \hookrightarrow \text{RV} \setminus \{0\} \twoheadrightarrow \Gamma$  (such a splitting corresponds to an angular component map  $K \rightarrow k$  of the valued field); then it becomes interdefinable with the language  $\mathcal{L}'$  consisting of  $k$  with the ring language and  $\Gamma$  with the language  $\{0, +, -, <\}$  of ordered abelian groups (where  $\text{RV}$  is identified with  $k \times \Gamma$ ). In particular, any set  $X \subseteq k^n \times \Gamma^m$  definable in our original language is also  $\mathcal{L}'$ -definable. Since  $\mathcal{L}'$  contains no connection between  $k$  and  $\Gamma$ , the Proposition follows.  $\square$

At some point, we will use the following easy consequence of the above hypothesis:

**Remark 6.3.** For any parameter set  $A \subseteq k$ , we have  $\text{acl}(A) \cap \Gamma = \text{acl}(\emptyset) \cap \Gamma$  (where  $\text{acl}$  is the algebraic closure in the model theoretic sense). Using the order on  $\Gamma$ , we get the same with  $\text{acl}$  replaced by the definable closure. In particular, if  $\mathcal{L} = \mathcal{L}_{\text{Hen}}$  and  $K$  is either real closed or algebraically closed, then  $\Gamma$  is a pure divisible ordered abelian group and the only finite,  $A$ -definable subsets of  $\Gamma$  are  $\emptyset$  and  $\{0\}$ .

**6.1. An analogue of Whitney's Condition (b).** Our main theorem about the existence of  $t$ -stratifications only speaks about the dimension of translatability spaces. The following theorem additionally (partially) specifies their direction. The analogue of Whitney's Condition (b) will then be a corollary.

**Theorem 6.4.** *Suppose that the language  $\mathcal{L}$  and the  $\mathcal{L}$ -theory  $\mathcal{T}$  satisfy Hypotheses 2.8 and 6.1 and that  $K$  is a model of  $\mathcal{T}$ . Let  $\chi: K^n \rightarrow \text{RV}^{\text{eq}}$  be a coloring and let  $x \in K^n$  be any point. Let  $\Xi \subseteq \text{RV}^{(n)} \setminus \{0\}$  be the set of those  $\xi$  such that  $\chi$  is not  $\text{dir}_{\text{RV}}(\xi)$ -translatable on the ball  $B := x + \text{rv}^{-1}(\xi)$ . Then  $v_{\text{RV}}(\Xi)$  is finite.*

**Remark 6.5.** Note that in general,  $\Xi$  is not definable. However, we can choose a  $t$ -stratification  $(S_i)_i$  reflecting  $\chi$  and refine  $\chi$  to  $((S_i)_i, \chi)$ ; after this modification,  $\Xi$  is definable (over  $x$  and the parameters of the original  $\chi$ ) by Lemma 3.15.

*Proof of Theorem 6.4.* By Remark 6.5, we may assume that  $\Xi$  is definable. Without loss, fix  $x = 0$  and suppose for contradiction that  $v_{\text{RV}}(\Xi)$  is infinite. By orthogonality of the value group and the residue field, there exists a one-dimensional  $V \subseteq k^n$  such that the subset  $\Xi_0 := \{\xi \in \Xi \mid \text{dir}_{\text{RV}}(\xi) \in V\}$  is already infinite. Choose a lift  $\tilde{V} \subseteq K^n$  of  $V$  and consider the coloring  $\chi' := (\chi, \tilde{V})$ . For  $\xi \in \Xi_0$ ,  $\chi$  is not  $V$ -translatable on the ball  $B := \text{rv}^{-1}(\xi)$ ; on the other hand,  $\tilde{V} \cap B \neq \emptyset$ , so  $\text{tsp}_B(\tilde{V}) = V$ , which implies that  $\chi'$  is not translatable at all on  $\text{rv}^{-1}(\xi)$ . In particular, if  $(S_i)_i$  is a  $t$ -stratification reflecting  $\chi'$ , then we have  $\text{rv}^{-1}(\xi) \cap S_0 \neq \emptyset$  for all  $\xi \in \Xi_0$ , contradicting finiteness of  $S_0$ .  $\square$

In the classical version of Whitney's Condition (b), one has two sequences of points in two different strata  $S_d$  and  $S_j$  with  $d < j$ , and both sequences converge to the same point in  $S_d$ . In the valued field version, each sequence is replaced by a single point, and “converging to the same point in  $S_d$ ” is replaced by “lying in a common ball  $B \subseteq S_{\geq d}$ ”. In the proof of Proposition 6.10, we will see how this implies the classical Condition (b) via non-standard analysis.

**Corollary 6.6.** *Let  $(S_i)_i$  be a  $\emptyset$ -definable  $t$ -stratification of a  $\emptyset$ -definable ball  $B_0 \subseteq K^n$ , let  $B \subseteq B_0$  be a sub-ball, and let  $d$  be maximal with  $B \subseteq S_{\geq d}$ . Then there exists a finite  $\ulcorner B \urcorner$ -definable set  $M \subseteq \Gamma$  such that the following holds. For any  $j > d$ , any  $x' \in B \cap S_d$ , and any  $y' \in B \cap S_j$ , if  $v(x' - y') \notin M$ , then  $\text{dir}(x' - y') \in \text{tsp}_{B'}((S_i)_i)$ , where  $B' \subseteq S_{\geq j}$  is a ball containing  $y'$ .*

*Proof.* Let  $\pi: B \rightarrow K^d$  be an exhibition of  $\text{tsp}_B((S_i)_i)$ . Choose any  $z \in \pi(B)$ , consider a point  $x \in \pi^{-1}(z) \cap S_d$ , and apply Theorem 6.4 to  $(S_i)_i$  (interpreted as a coloring) and  $x$ . This yields a finite,  $x$ -definable set  $v_{\text{RV}}(\Xi) \subseteq \Gamma$ ; doing this for all  $x \in \pi^{-1}(z) \cap S_d$  and taking the union of the (finitely many) corresponding sets  $v_{\text{RV}}(\Xi)$  yields a finite,  $z$ -definable set which we denote by  $M_z$ . Now for any other  $z' \in \pi(B)$ , Lemma 3.7 yields a risometry  $\alpha: B \rightarrow B$  sending  $\pi^{-1}(z)$  to

$\pi^{-1}(z')$ ; extending  $\alpha$  by the identity on  $B_0 \setminus B$ , we get a risometry which shows that  $M_z = M_{z'}$ ; hence  $M := M_z$  is  $\ulcorner B \urcorner$ -definable.

Now let  $x' \in B \cap S_d$  and  $y' \in B \cap S_j$  be given with  $v(x' - y') \notin M$  and set  $z = \pi(x')$ . Since  $\lambda := v(y' - x') \notin M_z$ ,  $(S_i)_i$  is  $\text{dir}(x' - y')$ -translatable on  $B_1 := B(y', > \lambda)$  and hence also on  $B' \subseteq B_1$ .  $\square$

**6.2. The classical Whitney conditions.** Let us recall the definition of Whitney stratifications; see e.g. [1] for more details. We will consider Whitney stratifications both over  $k = \mathbb{R}$  and  $k = \mathbb{C}$ , in a semi-algebraic resp. algebraic setting. A Whitney stratification is a partition of  $k^n$  into certain kinds of manifolds. In the case  $k = \mathbb{R}$ , we will work with Nash manifolds and also with a weakening of that notion.

**Definition 6.7.** A *Nash manifold* is a  $C^\infty$ -sub-manifold of  $\mathbb{R}^n$  (for some  $n$ ), which is  $\mathcal{L}_{\text{ring}}$ -definable (or, equivalently, which is semi-algebraic). By a  $C^1$ -*Nash manifold*, we mean a  $C^1$ -sub-manifold of  $\mathbb{R}^n$  which is  $\mathcal{L}_{\text{ring}}$ -definable.

Note that by “ $M$  is a sub-manifold of  $k^n$ ” we mean that also the inclusion map  $M \hookrightarrow k^n$  is in the corresponding category, i.e., either  $C^1$  or  $C^\infty$  (but we do not require  $M$  to be closed in  $k^n$ ). All our manifolds will be sub-manifolds of some  $k^n$  in this sense (for  $k$  either  $\mathbb{R}$  or  $\mathbb{C}$ ); this will not always be written explicitly.

In the case  $k = \mathbb{C}$ , we will only have one notion of manifolds, namely algebraic sub-manifolds of  $\mathbb{C}^n$ . Note that this is in perfect analogy to the case  $k = \mathbb{R}$ : if we simply replace  $\mathbb{R}$  by  $\mathbb{C}$  in Definition 6.7, then “definable” means constructible instead of semi-algebraic; moreover “differentiable” should now be read as “complex differentiable”. Thus in that case, both kinds of manifolds introduced in Definition 6.7 simply become algebraic manifolds.

In the remainder of the section, we will treat  $k = \mathbb{R}$  and  $k = \mathbb{C}$  simultaneously, and we will write “Nash/algebraic manifolds” or “ $C^1$ -Nash/algebraic manifolds” (depending on the notion of manifold we want to consider in the case  $k = \mathbb{R}$ ).

We will not require our manifolds to be connected, but if they are not, then each connected component has to have the same dimension.

For a  $C^1$ -Nash/algebraic manifold  $M \subseteq k^n$  and a point  $x \in M$ , there is a well-defined notion of tangent space  $T_x M \subseteq k^n$  of  $M$  at  $x$ . Such a space can be seen as an element of the corresponding Grassmanian  $\mathbb{G}_{n, \dim M}(k)$  and as such, it makes sense to speak of limits of sequences of such spaces.

**Definition 6.8.** Let  $k$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . A *Whitney stratification* of  $k^n$  is a partition of  $k^n$  into Nash/algebraic manifolds  $(S_i)_{0 \leq i \leq n}$  with the following properties. (As always, we write  $S_{\leq i}$  for  $S_0 \cup \dots \cup S_i$ .)

- (1) For each  $i$ , either  $\dim S_i = i$  or  $S_i = \emptyset$ .
- (2) Each set  $S_{\leq i}$  is topologically closed in the analytic topology.
- (3) Each pair  $S_i, S_j$  with  $i < j$  satisfies *Whitney's Condition (a)*, i.e., for any element  $u \in S_i$  and any sequence  $v_\mu \in S_j$  converging to  $u$ , if  $\lim_{\mu \rightarrow \infty} T_{v_\mu} S_j$  exists, then

$$T_u S_i \subseteq \lim_{\mu \rightarrow \infty} T_{v_\mu} S_j.$$

- (4) Each pair  $S_i, S_j$  with  $i < j$  satisfies *Whitney's Condition (b)*, i.e., for any two sequences  $u_\mu \in S_i, v_\mu \in S_j$  both converging to the same element  $u \in S_i$ , if both  $\lim_{\mu \rightarrow \infty} T_{v_\mu} S_j$  and  $\lim_{\mu \rightarrow \infty} k \cdot (u_\mu - v_\mu)$  exist, then

$$\lim_{\mu \rightarrow \infty} k \cdot (u_\mu - v_\mu) \subseteq \lim_{\mu \rightarrow \infty} T_{v_\mu} S_j.$$

We will say that  $(S_i)_i$  is a  $C^1$ -Whitney stratification if it is a partition of  $k^n$  into  $C^1$ -Nash/algebraic manifolds satisfying the above conditions (1) – (4).

Often, one additionally requires that the topological closure of any connected component of any  $S_j$  is the union of some connected components of some of the  $S_i$ ,  $i < j$ . However, once one knows how to obtain Whitney stratifications in our sense, it is easy to also obtain this additional condition.

**6.3. Transfer to the Archimedean case.** Let  $k$  be either  $\mathbb{R}$  or  $\mathbb{C}$ . We will consider  $k$  as a structure in the language  $\mathcal{L}_{\text{absring}} := \mathcal{L}_{\text{ring}} \cup \{|\cdot|\}$ , where  $|\cdot|: k \rightarrow \mathbb{R}_{\geq 0} \subseteq k$  is the absolute value. (Of course, in the case  $k = \mathbb{R}$ ,  $|\cdot|$  is already  $\mathcal{L}_{\text{ring}}$ -definable.) Fix  $K$  to be a (non-principal) ultra-power of  $k$  with index set  $\mathbb{N}$ ; this will be the non-standard model of  $k$  we will be working in. (In fact, any  $\aleph_1$ -saturated elementary extension of the  $\mathcal{L}_{\text{absring}}$ -structure  $k$  would do; however, in the following we will use notation closer to non-standard analysis than to “classical” model theory.)

Denote the canonical embedding  $k \hookrightarrow K$  by  $u \mapsto {}^*u$  and denote the image of  $k$  under this map by  $K_0$ . For any set  $X \subseteq k^n$ , the ultra-power of  $X$ , considered as a subset of  $K^n$ , will be denoted by  ${}^*X$ . (In particular,  ${}^*k = K$  and  ${}^*\mathbb{R} \subseteq K$ .)

Define

$$\mathcal{O}_K := \{x \in K \mid \exists(u \in \mathbb{R}) |x| < u\};$$

this is a valuation ring, turning  $K$  into a valued field which is Henselian and of equi-characteristic 0. The maximal ideal is

$$\mathcal{M}_K = \{x \in K \mid \forall(u \in \mathbb{R}_{>0}) |x| < u\},$$

the residue field is  $k$ , and  $\text{res}: \mathcal{O}_K \rightarrow k$  is simply the standard part map.

Using the absolute value on  $k$ , we can define the Euclidean norm on  $k^n$ , which we denote by  $|\cdot|_2: k^n \rightarrow \mathbb{R}_{\geq 0}$ ; this also induces an “Euclidean norm”  $|\cdot|_2: K^n \rightarrow {}^*\mathbb{R}_{\geq 0}$ , and this Euclidean norm induces a topology on  $K^n$ , given by the subbase  $\{x \in K^n \mid |x - a|_2 < r\}$ ,  $a \in K^n$ ,  $r \in {}^*\mathbb{R}_{>0}$ . This topology is the same as the valuation topology on  $K^n$ , since for any  $a \in K^n$ , any  $\lambda \in \Gamma$ , and any  $r \in {}^*\mathbb{R}_{>0}$  with  $v(r) > \lambda$ , we have

$$B(a, > v(r)) \subseteq \{x \in K^n \mid |x - a|_2 < r\} \subseteq B(a, > \lambda);$$

note that we continue to use the notations  $B(a, > \lambda)$ ,  $B(a, \geq \lambda)$  for balls in the valuative sense.

Let  $X \subseteq k^n$  be any definable set. Any sequence  $(u_\mu)_{\mu \in \mathbb{N}}$  with  $u_\mu \in X$  and  $\lim_{\mu \rightarrow \infty} u_\mu = u \in k^n$  represents an element of  $\text{res}^{-1}(u) \cap {}^*X$  in the ultra-product; vice versa, any element of  $\text{res}^{-1}(u) \cap {}^*X$  can be represented by a sequence in  $X$  converging to  $u$ . If  $(u_\mu)_\mu$  is such a converging sequence, we will write  $[u_\mu]$  for the corresponding element of  $\text{res}^{-1}(\lim_\mu u_\mu)$ . We will also use this notation with more complicated expressions inside the square brackets; the index variable of the sequence will always be  $\mu$ . Note that square brackets commute with definable maps: if, in addition to  $X$  and  $u_\mu$  as above, we have definable  $Y \subseteq k^m$  and  $f: X \rightarrow Y$ , then  $[f(u_\mu)] = {}^*f([u_\mu])$ , where  ${}^*f$  denotes the corresponding map  ${}^*X \rightarrow {}^*Y$ .

The following lemma is almost trivial, but it is the main tool which makes the transfer between  $k$  and  $K$  work. (Note that we implicitly use that the two different topologies on  $K$  coincide.)

**Lemma 6.9.** *For any definable set  $X \subseteq k^n$  and any element  $u \in k^n$ , the following are equivalent:*

- (1)  $u$  lies in the topological closure of  $X$ ;
- (2)  $*u$  lies in the topological closure of  $*X$ ;
- (3)  $*X \cap \text{res}^{-1}(u)$  is non-empty.

*Proof.* (1)  $\iff$  (2) follows from definability of being in the topological closure. (1)  $\iff$  (3): If  $u$  lies in the closure of  $X$ , then any sequence  $v_\mu \in X$  converging to  $u$  yields an element  $[v_\mu] \in X \cap \text{res}^{-1}(u)$ . Vice versa, if  $[v_\mu] \in X \cap \text{res}^{-1}(u)$ , then we may assume  $v_\mu \in X$  for all  $\mu$  and thus  $u = \lim_\mu v_\mu$  lies in the closure of  $X$ .  $\square$

Note that the equivalence (2)  $\iff$  (3) does not hold if one replaces  $*X$  by an arbitrary definable subset of  $K^n$ ; the point is that  $*X$  is  $\mathcal{L}_{\text{absring}}$ -definable and using only parameters from  $K_0$ .

We will need to apply Lemma 6.9 to the Grassmanian  $\mathbb{G}_{n,d}$  (for some  $d, n \in \mathbb{N}, d \leq n$ ). Indeed,  $\mathbb{G}_{n,d}(k)$  can be seen as a definable set in  $k$  and we have  $*(\mathbb{G}_{n,d}(k)) = \mathbb{G}_{n,d}(K)$ . Moreover, since  $\mathbb{G}_{n,d}$  is a projective variety, we have a map  $\text{res}: \mathbb{G}_{n,d}(K) \rightarrow \mathbb{G}_{n,d}(k)$  defined on the whole of  $\mathbb{G}_{n,d}(K)$ . For a  $d$ -dimensional subspace  $V \subseteq K^n$ , “ $\text{res}(V)$ ” now has two different meanings: we may view  $V$  as an element of  $\mathbb{G}_{n,d}(K)$ , apply  $\text{res}$ , and consider the corresponding subspace of  $k^n$ , or we may apply  $\text{res}$  directly to the elements of  $V$  (as in Definition 2.4). One easily checks that those two meanings of “ $\text{res}(V)$ ” coincide.

Now we can formulate the main proposition of this section.

**Proposition 6.10.** *Suppose that  $(S_i)_i$  are  $\mathcal{L}_{\text{ring}}$ -definable subsets of  $k^n$  such that  $(*S_i)_i$  is a  $t$ -stratification of  $K^n$ . Then  $(S_i)_i$  is a  $C^1$ -Whitney stratification of  $k^n$  (in the sense of Definition 6.8).*

*Proof.* In this proof, we will use the letters  $u, v$  for elements of  $k^n$  and  $x, x'$  for elements of  $K^n$ . We have to prove conditions (1) – (4) of Definition 6.8 and that each  $S_i$  is a  $C^1$ -Nash/algebraic manifold.

Since dimension is definable, (1) follows from the corresponding property of  $*S_i$ . (To obtain a definition of dimension which works both in  $k^n$  and in  $K^n$ , we can replace the valuative ball in Definition 2.13 by a Euclidean ball.)

Using Lemma 6.9, closedness of  $*S_{\leq i}$  implies (2) and moreover that for any  $u \in S_d$ ,  $\text{res}^{-1}(u)$  is entirely contained in  $*S_{\geq d}$ ; in particular,  $(*S_i)_i$  is  $d$ -translatable on  $\text{res}^{-1}(u)$ .

Fix  $u \in S_d$  and set  $B := \text{res}^{-1}(u)$  and  $V_u := \text{tsp}_B((S_i)_i)$ . We claim that

$$(\diamond) \quad \text{affdir}(B \cap *S_d) \subseteq V_u,$$

i.e., for any  $x, x' \in B \cap *S_d$ , we have  $\text{dir}(x - x') \in V_u$ . (For dimension reasons, we then get  $\text{affdir}(B \cap *S_d) = V_u$ .) To prove this claim, choose an exhibition  $\pi: K^n \twoheadrightarrow K^d$  of  $V_u$ , set  $F := \pi^{-1}(\pi(*u)) \cap *S_d$ , and apply Lemma 6.9 to  $F \setminus \{*u\}$ . Since  $*u$  does not lie in the closure of  $F \setminus \{*u\}$ , we obtain  $F \cap B = \{*u\}$ , i.e.,  $\pi$ -fibers of  $*S_d$  in  $B$  consist of a single element. Now  $V_u$ -translatability of  $*S_d$  on  $B$  implies the claim.

Next, we prove that  $S_d$  is a  $C^1$ -manifold and that  $V_u$  is the tangent space at  $u$  for every  $u \in S_d$  and  $V_u$  as above. First note that each point  $u' \in S_d$  has a neighborhood  $B \subseteq k^n$  such that for a suitable coordinate projection  $\bar{\pi}: k^n \twoheadrightarrow k^d$ ,  $\bar{\pi}$  induces a bijection  $B \cap S_d \rightarrow \bar{\pi}(B)$ . (Indeed, this is a first order statement and it holds in  $K$ .) We will use  $\bar{\pi}$  as a chart of  $S_d$  around  $u'$ . To get that its inverse  $(\bar{\pi}|_{B \cap S_d})^{-1}$  is  $C^1$  and that  $V_u$  is the tangent space at  $u$  (for every  $u \in S_d$ ), it suffices to prove the following. For any  $u \in S_d$  and any two sequences  $v_\mu, v'_\mu \in S_d$  with  $\lim_\mu v_\mu = \lim_\mu v'_\mu = u$  and  $v_\mu \neq v'_\mu$ , if  $\lim_\mu k \cdot (v_\mu - v'_\mu)$  exists (in  $\mathbb{G}_{n,1}(k)$ ), then

$\lim_{\mu} k \cdot (v_{\mu} - v'_{\mu}) \subseteq V_u$ . So suppose that such  $u, v_{\mu}, v'_{\mu}$  are given. Working in  $\mathbb{G}_{n,1}$ , we have

$$\lim_{\mu} k \cdot (v_{\mu} - v'_{\mu}) = \text{res}([k \cdot (v_{\mu} - v'_{\mu})]) = \text{res}(K \cdot ([v_{\mu}] - [v'_{\mu}])).$$

Now  $[v_{\mu}], [v'_{\mu}] \in \text{res}^{-1}(u) \cap {}^*S_d$  implies  $\text{dir}([v_{\mu}] - [v'_{\mu}]) \in V_u$  by  $(\diamond)$  and hence  $\text{res}(K \cdot ([v_{\mu}] - [v'_{\mu}])) \subseteq V_u$ .

In the case  $k = \mathbb{C}$ , we just proved that  $S_d$  is  $C^1$  in the sense of complex differentiation, so in that case, we obtain that  $S_d$  is an algebraic manifold.

Sending a point  $u \in S_d$  to its tangent space  $T_u S_d$  is a definable map  $S_d \rightarrow \mathbb{G}_{n,d}(k)$ ; transferring this to  $K$  yields a notion of tangent space of  ${}^*S_d$  at any  $x \in {}^*S_d$ ; we denote that tangent space (which is a sub-space of  $K^n$ ) by  $T_x {}^*S_d$ . Fix  $x \in {}^*S_d$ . By definition, if  $x' \in {}^*S_d \setminus \{x\}$  is close to  $x$ , then  $K \cdot (x - x')$  is close to a space contained in  $T_x {}^*S_d$ . In particular and more precisely, there exists a ball  $B' \subseteq K^n$  containing  $x$  such that for any  $x' \in B' \cap {}^*S_d \setminus \{x\}$ ,  $\text{res}(K \cdot (x' - x)) \subseteq \text{res}(T_x {}^*S_d)$ . After possibly further shrinking  $B'$ ,  $(S_i)_i$  becomes  $d$ -translatable on  $B'$  and then, any one-dimensional subspace of  $\text{tsp}_{B'}((S_i)_i)$  is of the form  $\text{res}(K \cdot (x' - x))$  for some  $x' \in B' \cap {}^*S_d \setminus \{x\}$ . For dimension reasons, this implies

$$\text{res}(T_x {}^*S_d) = \text{tsp}_{B'}((S_i)_i).$$

Now consider Whitney's Condition (a), i.e., suppose we are given a point  $u \in S_d$  and a sequence  $v_{\mu} \in S_j$  ( $j > d$ ) as in Definition 6.8 (3). Set  $B := \text{res}^{-1}(u)$  and let  $B' \subseteq B \cap {}^*S_{\geq j}$  be a ball containing  $[v_{\mu}]$ . Then

$$\lim_{\mu} T_{v_{\mu}} S_j = \text{res}([T_{v_{\mu}} S_j]) = \text{res}(T_{[v_{\mu}]} {}^*S_j) = \text{tsp}_{B'}((S_i)_i) \supseteq \text{tsp}_B((S_i)_i) = T_u S_d.$$

For Whitney's Condition (b), suppose we are given  $u \in S_d$  and sequences  $u_{\mu} \in S_d$  and  $v_{\mu} \in S_j$  ( $j > d$ ) as in Definition 6.8 (4). Again set  $B := \text{res}^{-1}(u)$ . Since  $[u_{\mu}], [v_{\mu}] \in B \subseteq {}^*S_{\geq d}$ , we can apply Corollary 6.6 to  $[u_{\mu}] \in {}^*S_d$  and  $[v_{\mu}] \in {}^*S_j$  and get a finite,  $\mathcal{L}_{\text{Hen}}(\ulcorner B \urcorner)$ -definable set  $M \subseteq \Gamma$  such that  $v([u_{\mu}] - [v_{\mu}]) \notin M$  implies  $\text{dir}([u_{\mu}] - [v_{\mu}]) \in \text{tsp}_{B'}((S_i)_i)$  for a ball  $B' \subseteq {}^*S_{\geq j}$  containing  $[v_{\mu}]$ . In particular,  $M$  is  $\mathcal{L}_{\text{Hen}}(u)$ -definable (viewing  $u$  as an element of the residue field), so  $M \subseteq \{0\}$  by Remark 6.3 and thus indeed  $v([u_{\mu}] - [v_{\mu}]) \notin M$ . Therefore we obtain

$$\begin{aligned} \lim_{\mu} k \cdot (u_{\mu} - v_{\mu}) &= \text{res}([k \cdot (u_{\mu} - v_{\mu})]) = \text{res}(K \cdot ([u_{\mu}] - [v_{\mu}])) \\ &\subseteq \text{tsp}_{B'}((S_i)_i) = \text{res}(T_{[v_{\mu}]} {}^*S_j) = \text{res}([T_{v_{\mu}} S_j]) = \lim_{\mu} T_{v_{\mu}} S_j, \end{aligned}$$

which finishes the proof.  $\square$

Using the proposition, it is now easy to deduce that  $t$ -stratifications “are” also classical Whitney stratifications. To be consistent with Subsection 5.2, we fix an integral domain  $A$  of characteristic 0, we set  $\mathcal{L} := \mathcal{L}_{\text{Hen}}(A)$ , and we let  $\mathcal{T}$  be the theory of Henselian valued fields  $K$  of equi-characteristic 0 with ring homomorphism  $A \rightarrow K$ .

**Theorem 6.11.** *Let  $A$ ,  $\mathcal{L}$ , and  $\mathcal{T}$  be as defined right above. Suppose that  $\phi_{\nu}$  ( $\nu = 1, \dots, \ell$ ) and  $\psi_i$  ( $i = 0, \dots, n$ ) are  $\mathcal{L}_{\text{ring}}(A)$ -formulas in  $n$  free variables such that for any model  $K \models \mathcal{T}$ ,  $(\psi_i(K))_i$  is a  $t$ -stratification of  $K^n$  reflecting the coloring given by  $(\phi_{\nu}(K))_{\nu}$ . Suppose moreover that the formulas  $\psi_i$  are quantifier free. Then for both  $k = \mathbb{R}$  and  $k = \mathbb{C}$  and for any ring homomorphism  $A \rightarrow k$ , we have the following, where  $X_{\nu} := \phi_{\nu}(k)$  and  $S_i := \psi_i(k)$ .*

- (1)  $(S_i)_i$  is a Whitney stratification of  $k^n$  (see Definition 6.8)
- (2) Each  $X_\nu$  is a union of some of the connected components of the sets  $S_i$  (in the analytic topology).

In particular, each  $\psi_i$  is an algebraic variety which is smooth over the fraction field of  $A$ .

Note that by Corollary 5.9, for any  $\phi_\nu$  as above we can find  $\psi_i$  as above, so indeed we obtain a new proof of the existence of Whitney stratifications (for  $\mathcal{L}_{\text{ring}}(A)$ -definable subsets of  $\mathbb{R}^n$  or  $\mathbb{C}^n$ ).

*Proof of Theorem 6.11.* Let  $K$  be the non-standard model of  $k$  used in Proposition 6.10; we consider it as an  $\mathcal{L}$ -structure using the ring homomorphism  $A \rightarrow k \hookrightarrow K$ . Then the conclusion of Proposition 6.10 is that  $(S_i)_i$  is a  $C^1$ -Whitney stratification. To finish the proof of (1), we have to get rid of this “ $C^1$ ”. By taking  $k = \mathbb{C}$ , we obtain that each  $\psi_i(\mathbb{C})$  is an algebraic manifold; since  $\psi_i$  is quantifier free, it can be viewed as a variety which is smooth over  $\mathbb{C}$ ; in particular  $\psi_i(\mathbb{R})$  is a  $C^\infty$ -sub-manifold of  $\mathbb{R}^n$ .

It remains to verify (2). We have to show that for each  $d$  and each  $\nu$ , both  $S_d \cap X_\nu$  and  $S_d \setminus X_\nu$  are open in  $S_d$ . Since this is first order, we can instead prove the corresponding statement in  $K^n$ , i.e., that  ${}^*S_d \cap {}^*X_\nu$  and  ${}^*S_d \setminus {}^*X_\nu$  are open in  ${}^*S_d$ .

Let  $x \in {}^*S_d$  be given. We choose a ball  $B \subseteq {}^*S_{\geq d}$  containing  $x$ , we choose an exhibition  $\pi: B \rightarrow K^d$  of  $V := \text{tsp}_B((S_i)_i)$ , and we shrink  $B$  such that each  $\pi$ -fiber intersects  $B \cap {}^*S_d$  in a single point. Then  $V$ -translatability implies that the set  $B \cap {}^*S_d$  is either disjoint from  ${}^*X_\nu$  or entirely contained in  ${}^*X_\nu$ .  $\square$

## 7. SETS UP TO ISOMETRY IN $\mathbb{Q}_p$

The main conjecture of [8] was a description of certain trees  $T(X)$  associated to definable sets  $X \subseteq \mathbb{Z}_p^n$  (in the pure valued field language). The original motivation for the present article was to prove that main conjecture for big  $p$ , and indeed, the existence of  $t$ -stratifications can be seen as a better (and stronger) reformulation of that main conjecture. More precisely, we will prove the following.

**Theorem 7.1.** *For every  $\mathcal{L}_{\text{Hen}}$ -formula  $\phi(x)$  defining a subset of  $\mathcal{O}_K^n$  (for any valued field  $K$ ), there exists an  $N \in \mathbb{N}$  such that for all  $p > N$ , [8, Conjecture 1.1] holds for  $X := \phi(\mathbb{Q}_p)$ , i.e., “the tree  $T(X)$  associated to  $X$  is of level  $\leq \dim X$ ”.*

The definition of “tree of level  $d$ ” from [8] is very long and technical, so I will not repeat it here. Let me only recall the definition of  $T(X)$ . (In [8], trees were considered as directed graphs; here, we consider them as partially ordered sets, which is more handy but which doesn’t make a real difference.)

**Definition 7.2.** For a set  $X \subseteq \mathbb{Z}_p^n$ , the tree  $T(X)$  associated to  $X$  is the partially ordered set of those balls  $B \subseteq \mathbb{Z}_p^n$  which intersect  $X$  non-trivially. The ordering is given by inclusion. (If  $X \neq \emptyset$ , then  $T(X)$  is a tree whose root is  $\mathbb{Z}_p^n$ .)

(From the point of view of the present article, there is no reason to restrict to  $\mathbb{Z}_p^n$ ; one could as well consider  $X \subseteq \mathbb{Q}_p^n$  and allow  $B \subseteq \mathbb{Q}_p^n$  in the definition of  $T(X)$ .)

To get some intuition, recall that by [8, Lemma 3.1], for any two subsets  $X, Y \subseteq \mathbb{Z}_p^n$ , we have a canonical bijection between the set of isomorphisms  $T(X) \rightarrow T(Y)$  and the set of isometries  $\bar{X} \rightarrow \bar{Y}$ , where the bar denotes the  $p$ -adic closure (as

always, on  $\mathbb{Z}_p^n$  we use the ultra-metric induced by the maximum norm). This means that describing the tree of  $X$  is essentially the same as describing  $\bar{X}$  up to isometry. In [7] (in particular in Section 4.3), some more details are given on how to translate the condition “having a tree of level  $d$ ” into a geometric condition involving isometries. This translation already has some resemblance with t-stratifications.

A remark about naming: what is called “level  $d$ ” in [8] is called “level  $\leq d$ ” in [7]. The latter naming makes more sense, since such trees correspond to definable sets of dimension  $\leq d$ ; we also use that latter naming in the present article.

*Proof of Theorem 7.1.* Let a definable set  $X \subseteq \mathbb{Z}_p^n$  be given. By Corollary 4.11, we find a t-stratification  $(S_i)_i$  reflecting  $X$ . From this, we have to deduce that  $T(X)$  is of level  $\leq \dim X$ .

As a skeleton for our tree  $T(X)$ , we can take  $T(S_0) \cap T(X)$ . (It has only finitely many bifurcations since  $S_0$  is finite.) If  $\dim X = 0$ , then  $X \subseteq S_0$  (by Lemma 5.1) and we are done. Otherwise, let  $\mathcal{B} \subseteq \text{RV}^{\text{eq}}$  be a set parametrizing all maximal balls  $B \subseteq \mathbb{Z}_p^n \setminus S_0$  and write  $B_\lambda$  for the ball corresponding to  $\lambda \in \mathcal{B}$ .

For any such  $B_\lambda$ , choose a one-dimensional sub-vector space  $V \subseteq \text{tsp}_B((S_i)_i)$ , and suppose without loss that the projection  $\pi: B_\lambda \rightarrow \mathbb{Q}_p$  to the first coordinate is an exhibition of  $V$ . By 1-translatibility, there is a risometry  $\phi: B_\lambda \rightarrow B_\lambda$  such that  $\phi^{-1}((S_i)_i, X)$  is  $\tilde{V}$ -translation invariant on  $B_\lambda$  for some one-dimensional  $\tilde{V} \subseteq \mathbb{Q}_p^n$ . By replacing  $\phi$  by  $\phi \circ \psi$  for some suitable isometry  $\psi: B_\lambda \rightarrow B_\lambda$ , we may assume that  $\phi^{-1}((S_i)_i, X)$  is  $(K \times \{0\}^{n-1})$ -translation invariant, at the cost that  $\phi$  now is only an isometry and not a risometry.

Choose a fiber  $F_\lambda = \pi^{-1}(x)$  ( $x \in \pi(B_\lambda)$ ); we may assume that  $\phi$  is the identity on  $F_\lambda$ . Since trees are preserved by isometries, the tree of  $X$  on  $B_\lambda$  is isomorphic to the tree of  $\phi^{-1}(X)$  on  $B_\lambda$ , which is equal to the product of the tree of  $X \cap F_\lambda$  with the tree of the ball  $\pi(B_\lambda)$ . Now repeat the whole process with the definable set  $X \cap F_\lambda$  (considered as a subset of  $F_\lambda \subseteq \mathbb{Z}_p^{n-1}$ ), using that by Lemma 3.16  $(S_i)_i$  induces a t-stratification on  $F_\lambda$  reflecting  $X \cap F_\lambda$ .

To finish the proof that the tree of  $X$  is of level  $\leq d$ , it remains to verify that the tree of  $X \cap F_\lambda$  varies uniformly with  $\lambda$  (in the sense of [8]). Let us show that its skeleton  $T(S_1 \cap F_\lambda) \cap T(X \cap F_\lambda) =: \mathcal{T}_\lambda$  varies uniformly with  $\lambda$ ; the same argument then also applies to iterated side branches.

For any enumeration  $(a_\mu)_\mu$  of  $S_1 \cap F_\lambda$ , we can consider the matrix  $(v(a_\mu - a_\nu))_{\mu, \nu}$ ; let  $M_\lambda$  be a code for the set of all those matrices corresponding to different enumerations. (Thus  $M_\lambda$  is a “matrix up to simultaneous permutation of the rows and columns”.) On the one hand,  $M_\lambda$  completely describes the skeleton  $\mathcal{T}_\lambda$ ; on the other hand,  $M_\lambda$  is determined by the risometry type of  $S_1 \cap F_\lambda$ , so it does not depend on the particular choice of the fiber  $F_\lambda$  inside  $B_\lambda$ , and thus, the map  $\lambda \mapsto M_\lambda$  is definable. This implies that we can find a finite partition of  $\mathcal{B}$  into definable subsets  $\mathcal{A}$  on each of which  $M_\lambda$  only depends on  $\text{rad}_c(B_\lambda)$  (this uses finiteness of the residue field). Since both,  $\text{rad}_c(B_\lambda)$  and  $M_\lambda$  live in the value group, the definable function sending  $\text{rad}_c(B_\lambda)$  to  $M_\lambda$  is piecewise linear, as required in [8].  $\square$

Note that we obtained a slightly stronger description of the structure of the trees than being of level  $\leq d$ . In terms of [8], we proved that the finite trees  $\mathcal{F}$  at the beginning of the side branches have only depth 1. The reason we get this stronger result is that we only consider large characteristics. The necessity of these finite trees in small characteristic can be translated into the language of t-stratifications,

which yields a good guess on the kind of t-stratifications one should expect in mixed characteristic; see Subsection 9.2.

In [8, Section 7], several strengthenings of the conjecture about the trees have been proposed; let us have a look at these strengthenings.

- The tree  $T(\mathbb{Z}_p^n)$  can be considered as an auxiliary sort; then, for any definable  $X \subseteq \mathbb{Z}_p^n$ , the tree  $T(X)$  is a definable subsets of  $T(\mathbb{Z}_p^n)$ . Conjecture 7.1 of [8] describes arbitrary definable subsets  $Y \subseteq T(\mathbb{Z}_p^n)$  instead of only those of the form  $T(X)$ .

For big  $p$ , it should also be possible to prove that conjecture using t-stratifications, by turning  $Y \subseteq T(\mathbb{Z}_p^n)$  into the following coloring of  $\mathbb{Z}_p^n$ :

$$\chi(x) := \ulcorner \{\gamma \in \Gamma \mid B(x, \geq \gamma) \in Y\} \urcorner.$$

Then we choose a stratification reflecting  $\chi$  and continue as in the proof of Theorem 7.1.

- In [8, Section 7.2], a version of the conjecture has been proposed for arbitrary Henselian valued fields of characteristic  $(0, 0)$  (without giving much details). Note first that in the proof of Theorem 7.1, at the place where we used finiteness of the residue field, we can use orthogonality of the residue field and the value group instead, so the proof also works in that case.

However, as observed in [8], a straightforward translation of [8, Conjecture 1.1] would not be very meaningful; to get a useful result, one has to give a more precise description of the trees, which involves some residue field formulas defining certain sets of children of certain nodes. From today's point of view, this is simply a first step from isometries to risometries. Thus driving the ideas of [8, Section 7.2] further, in the end, the characteristic  $(0, 0)$  conjecture would essentially turn into a conjecture about the existence of t-stratifications (which, by the way, is how the notion of t-stratification in the present article arose).

## 8. EXAMPLES

This section contains some example sets  $X \subseteq K^n$  and corresponding t-stratifications; the main goal of the examples is to show the difference to classical Whitney stratifications. Many computations are not carried out, but I hope that the reader can get some intuition. Note that a good intuition is obtained as follows. If an algebraic set in  $K^n$  is given, then think of the corresponding algebraic set in  $\mathbb{R}^n$ . Two points in  $K^n$  whose difference has strictly positive valuation correspond to two points in  $\mathbb{R}^n$  which are very close together, whereas negative valuation corresponds to being very far apart (the non-standard analysis point of view of Subsection 6.3 makes this precise).

The first example is model theoretic and uses some notation from Section 2; the second and third do not require any special knowledge.

**8.1. Subsets of  $K$ .** Suppose that  $X$  is a ball in  $K$ . Then on any ball  $B$  which is strictly bigger than  $X$ , we have no translatability, and hence such a  $B$  must contain an element of  $S_0$ . This could be achieved by putting any element of  $X$  into  $S_0$ , but in Theorem 4.10, we require  $S_0$  to be definable over the same parameter set as  $X$ , and  $\emptyset$ -definable balls  $X$  might contain no  $\emptyset$ -definable points. However, in that case, b-minimality (Definition 2.9) ensures that there exists a  $\emptyset$ -definable point  $x \in K$

such that  $X = x + \text{rv}^{-1}(\xi)$  for some  $\xi \in \text{RV}$ ; then any  $B$  strictly bigger than  $X$  contains  $x$  and thus, taking  $S_0 = \{x\}$  works.

**8.2. The parabola.** Consider the parabola  $X = \{(x, x^2) \mid x \in K\}$ . A first idea would be to take  $S_0 = \emptyset$ ,  $S_1 = X$  and  $S_2 = K^2 \setminus X$ . This works for small balls: intuitively, on a sufficiently small ball, the parabola is almost a straight line, which implies translatability in direction of the tangent space. (Here, balls of the size  $\mathcal{M}_K \times \mathcal{M}_K$  are already sufficiently small.)

However, the definition of t-stratification also requires translatability on big balls not intersecting  $S_0$ , which is a problem on the ball  $B_0 := \mathcal{O}_K^2$ . The solution is to add any element of  $B_0$  to  $S_0$ , e.g. the point  $(0, 0)$ . After this modification, it is not difficult to check that we indeed have a t-stratification: for balls containing  $(0, 0)$ , there is nothing to check, and neither for balls not intersecting  $X$ , so the only balls left are “big balls far above the origin”. There, one can verify that the parabola is sufficiently straight and sufficiently steep to get vertical translatability.

Having to add the point  $(0, 0)$  to  $S_0$  sounds quite counter-intuitive in this example; the next example explains a bit why this makes sense.

**8.3. Almost-singularities.** Consider the curve  $X = \{(x, y) \mid xy = a\}$  for some  $a \in K$ , and let us suppose that  $v(a)$  is big. Then  $X$  is smooth as a curve over  $K^2$ , but (thinking of the real picture), you have to look very closely to see that there is no singularity at  $(0, 0)$ . This kind of almost-singularity is seen by t-stratifications: we have no translatability on the closed ball  $B$  around  $(0, 0)$  of radius  $\frac{1}{2}v(a)$  (assuming  $\frac{1}{2}v(a) \in \Gamma$ ) and hence the point  $(0, 0)$  (or any other point sufficiently close to it) has to be put into  $S_0$ .

In the algebraic language and thinking of  $X$  as a scheme over  $\mathcal{O}_K$ , the generic fiber of  $X$  is smooth, but if  $v(a) > 0$ , then the special fiber of  $X$  has a singularity at  $(0, 0)$ ; this singularity is seen by the t-stratification.

## 9. OPEN QUESTIONS

There are several ways in which it might be possible to enhance the results of this article.

**9.1. A stronger notion of t-stratification.** Recall from the introduction that t-stratifications do not satisfy the classical Condition (a) of Whitney: For two strata  $S_d, S_j$  with  $d < j$ ,  $x \in S_d$ , and  $y \in S_j$ , we have that  $T_y S_j$  is “close to containing  $T_x S_d$  when  $y$  is close to  $x$ ”, whereas Condition (a) requires that  $T_y S_j$  converges to a space containing  $T_x S_d$ . Also recall that in the  $p$ -adics, the existence of Whitney stratifications in this classical sense has been proven in [2]. It seems plausible that there exists a common generalization of both kinds of stratifications (at least in equi-characteristic 0 and if the valuation is of rank one). Such a generalization might be defined as follows.

Let us define “stronger risometries”: maps  $\phi$  such that  $v((\phi(x) - \phi(x')) - (x - x')) > v(x - x') + \delta$  for some given  $\delta \geq 0$ . (For  $\delta = 0$ , this is just a usual risometry.) This yields a stronger notion of translatability; let us call it “ $\delta$ -strong translatability”.

Using this notion, a “strong t-stratification” should roughly require that for any  $\delta \geq 0$  and any ball  $B$  “sufficiently far away from  $S_{\leq d-1}$ ”, we have  $\delta$ -strong  $d$ -translatability on  $B$ . The question is what “sufficiently far away from  $S_{\leq d-1}$ ” should mean exactly. In usual t-stratifications, we only required  $B \cap S_{\leq d-1} = \emptyset$ ,

but this does not work for  $\delta > 0$ . The exact condition could be something like: for any ball  $B'$  contained in  $S_{\geq d}$  and any  $\delta \geq 0$ , there exists a  $\delta' \geq 0$  such that we have  $\delta$ -strong  $d$ -translatability on any subball  $B$  of  $B'$  with  $\text{rad}_o B \leq \text{rad}_o B' - \delta'$ . (Maybe one can also be more specific on how  $\delta'$  depends on  $\delta$ .)

Note that this indeed implies Condition (a). For any  $x \in S_d$  and any  $\delta \geq 0$ , there exists a ball  $B$  around  $x$  which is sufficiently far away from  $S_{\leq d-1}$  in the above sense, and  $\delta$ -strong translatability on  $B$  then implies that for  $y \in B \cap S_j$ ,  $T_y S_j$  has at most (valuative) distance  $\delta$  from a space containing  $T_x S_d$ .

**9.2. Mixed characteristic.** It should be possible to prove the existence of a variant of t-stratifications in mixed characteristic, but again, it is not entirely clear how this variant has to be formulated. For a ball  $B \subseteq S_{\geq d}$ , even 0-strong (i.e., usual)  $d$ -translatability can only be expected on subballs  $B'$  of  $B$  with  $\text{rad}_o B' \leq \text{rad}_o B - \delta$  for some fixed  $\delta$  (depending only on the t-stratification). This can be seen, for example, at the cusp curve in characteristic 2. In terms of the description of the trees of [8], this  $\delta$  would be exactly the maximal length of the finite trees appearing at the beginning of side branches; see the comment after the proof of Theorem 7.1.

When the value of the residue characteristic  $p$  is finite (i.e., when there are only finitely elements of  $\Gamma$  between 0 and  $v(p)$ ), then in the previous paragraph, it should be also possible to require  $\delta$  to be finite, and it seems to me that the resulting notion of t-stratification is the “right one”. If  $v(p)$  is not finite, then we are probably forced to allow finite multiples of  $v(p)$  for  $\delta$ , but I am afraid that then the conjecture becomes “too weak”, in the sense that there are still important things which can (and should) be said about balls  $B'$  which are only slightly smaller than  $B$ .

**9.3. Getting classical Whitney stratifications more generally.** The fact that the existence of t-stratifications implies the existence of Whitney stratifications should also work in other languages than the pure (semi-)algebraic one. For this to work, we need the existence of t-stratifications  $(S_i)_i$  which are defined without using the valuation. Probably Proposition 5.6 can be applied to prove such a result, but I did not check it. In the algebraic language, we used this to deduce a posteriori that each  $S_i$  is smooth. This too, should work more generally, again with an argument that manifolds in  $\mathbb{C}^n$  which are  $C^1$  in the sense of complex differentiation are automatically smooth.

**9.4. Minimal t-stratifications.** It would be nice if, for every definable set  $X \subseteq K^n$ , there would be a “minimal” t-stratification  $(S_i)_i$  reflecting  $X$ . “Minimal” could mean that for any other t-stratification  $(S'_i)_i$  reflecting  $X$ , we have  $S_{\leq i} \subseteq S'_{\leq i}$  for all  $i$ . Moreover, one might hope that for a minimal  $(S_i)_i$ , a definable isometry  $K^n \rightarrow K^n$  preserves  $X$  if and only if it preserves  $(S_i)_i$  (in general, there are less isometries preserving  $(S_i)_i$ ). In the case of Whitney stratifications of complex analytic spaces, minimal stratifications in the first sense have indeed been constructed by Teissier; see [9].

There are (at least) two reasons for minimal t-stratifications not to exist, but for both of them, there is some hope to overcome the problem. The first obstacle is that the non-localness of the conditions of t-stratifications sometimes forces us to choose any point in a ball which we have to put into a smaller stratum; see the examples in Section 8. It might be possible to overcome this problem as follows: instead of letting  $S_{\leq i}$  be a subset of  $K^n$ , we let it be a subset of all balls in  $K^n$ ,

where points are also considered as balls. Then we require  $d$ -translativity on a ball  $B \subseteq K^n$  iff no ball of  $S_{\leq d-1}$  is (strictly?) contained in  $B$ . At least for the examples of Section 8, this seems to solve the problem.

A second problem is that one can construct a set  $X$  such that whether  $X$  is  $d$ -translatable on some given ball is not a definable function of the ball. (This can easily be done using a sufficiently evil residue field, e.g.  $\mathbb{Q}$ .) Since for  $t$ -stratifications,  $d$ -translativity is always definable (Lemma 3.15), any  $t$ -stratification reflecting  $X$  will necessarily have less isometries than  $X$  preserving it. I do not think that it is possible to solve this problem in general, but it might be possible to find a good condition on the residue field which avoids the problem. Here is a candidate: for any definable function  $f: k^n \rightarrow k$ , there exists a definable function  $\tilde{f}: \mathcal{O}_K^n \rightarrow \mathcal{O}_K$  such that  $\text{res} \circ \tilde{f} = f \circ \text{res}$ .

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